When Geography Shapes Preferences: Redesigning Teacher Assignment in Italy*

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Abstract

We investigate Italy's centralized teacher assignment system where teachers can rank "geographical regions", leading to ties in their rank order lists (ROLs). We show that the way ties in teachers' ROLs are resolved in the current assignment mechanism systematically violates teachers' priority rights and results in justified envy. We propose a new mechanism, Deferred Acceptance with Hierarchical Choice (DA-HC), which is strategy-proof, eliminates justified envy, and Pareto improves over the benchmark deferred acceptance mechanism with simple tie-breaking (DA-STB). Using administrative data, we provide evidence that DA-HC can potentially bring significant welfare improvements over the benchmark DA-STB.

Keywords: Market Design, Teacher Assignment, Geography, Indifferences, Deferred Acceptance with Hierarchical Choice.

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1 Introduction

In many countries the assignment of teachers to teaching positions in the public school system is centralized.[1] In these labor markets, teachers are usually public employees subject to a stringent regulation of wages.[2] As a result, it is challenging for policymakers to use wages as an instrument to provide incentives. Instead, the opportunity for teachers to move to more desirable positions remains a crucial policy lever, which makes the design of a well-functioning teacher assignment system a fundamental objective for policymakers.

In Italy, the assignment of teachers to teaching positions in public schools has been centralized at least from the 1970s.[3] Every year, around 100,000 tenured teachers submit a rank order list (ROL) to express their preferences for moving to more desirable positions. Schools are administratively embedded into "geographical regions" (municipalities within districts within provinces), which impact the assignment system in two ways. First, teachers have the option to rank an entire region (municipality, district, or province), considering themselves indifferent among all schools within that region.[4] Second, alongside factors such as seniority, family reasons, and educational qualifications, the regions of teachers' current schools determine their priority rights at different schools.[5] This creates a priority-based assignment problem, where teachers express preferences including indifference classes structured around geographical regions, while schools have strict priority orderings.

A key fairness objective in teacher assignment is to respect teachers' priority rights by eliminating justified envy, ensuring that no teacher prefers another teacher's assigned school while having higher priority at that school.[6] This principle has been stated in several

[1]Besides Italy, some other examples are France (Combe et al., 2022a,b; Terrier, 2014), Germany (Klein and Baur, 2019), Portugal (Rodrigues et al., 2019; Tomás, 2017), Turkey (Dur and Kesten, 2019), Teach for America in Chicago (Davis, 2022), Peru (Bobba et al., 2021; Ederer, 2023), Ecuador (Elacqua et al., 2022, 2021), Mexico (Pereyra, 2013), Sao Paulo (Elacqua and Rosa, 2023; Rosa, 2019), Czech Republic and Slovakia (Cechlárová et al., 2016, 2015).

[2]For example, the salary scale for Italian public school teachers is set at the national level through an agreement between the government and teacher unions, with no bargaining at the individual level.

[3]The first systematic regulation is a Presidential Decree enacted in 1974 (*Decreto del Presidente della Repubblica 31 maggio 1974, n. 417*), containing an entire paragraph on teacher mobility's rules.

[4]For example, a teacher might rank a school as her first choice and the school's municipality as her second choice, with the interpretation that the teacher prefers the school to any other school in the municipality, and she is indifferent between all the other schools in the municipality. In Section 6, we provide evidence from application data that teachers frequently rank regions in practice.

[5]See Appendix B.2 for a complete list of criteria that determine priorities, and Appendix B.3 for a more thorough description of priorities' composition. Note that, differently from teacher preferences, school priorities are *strict*, since teacher age is eventually used as a tie-breaker.

[6]This is a common policy objective in priority-based assignment problems. See, among others, Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003).

judgements of the Italian Court, as an expression of a fundamental Constitutional principle of non-discrimination, meritocracy and equal treatment of teachers.[7] Nevertheless, justified envy instances resulting in court cases have been under constant scrutiny.[8] In a parliamentary audition in September 2016, the Minister of Education that year, Stefania Giannini, admitted that approximately 3,000 teachers (2.4% of applicants) were going to the court because of priority violations, while leaving the source of the problem unexplained.[9]

In this paper, we show that a flaw in the way ties in teachers' ROLs are resolved systematically results in justified envy. Upon receiving ROLs from teachers, the Italian Ministry of Education determines the outcome using a variant of the deferred acceptance (DA) algorithm (Gale and Shapley, 1962) with a distinctive feature, which we call *Deferred Acceptance with Hierarchical Priorities (DA-HP)*. When teachers rank a region, they apply to all schools in that region one by one following an official ordering of schools that is publicly available (*Bollettini Ufficiali*). Notably, when considering applications to a particular school, an applicant who ranks the school as part of a finer region is granted priority over an applicant who ranks the school as part of a coarser region, as long as the coarser region has at least one school with vacancy at that step of the algorithm. This introduces artificial priority rights within the assignment algorithm as specified in art. 6, par. 5 of the national collective bargaining agreement ("synthetic preference" refers to ranking a region in contrast to a single school):

> "*...Since with the synthetic preference all the schools included in the synthetic code are indifferently requested, the first school with an available place is assigned to the teacher who requested it with precise or more limited indication at a territorial level, albeit with a lower score and the teacher who has expressed the synthetic preference is assigned the next available school within the expressed synthetic preference.*"

The rationale behind introducing artificial priority rights is clear. Without this addition, the mechanism would reduce to the DA algorithm with simple tie-breaking (DA-STB), where ties

[7]See, for instance, Court of Salerno (Labor Section) Judgment n.336, 2020, regarding a teacher appealing for a position that is assigned to lower priority teachers.

[8]In compliance with the general principle of transparency, for each school, the list of assigned teachers along with their scores and other priority rights are published online (see art. 6, par. 2, *Ordinanza Ministeriale*), so that applicants may check whether their priority rights are respected.

[9]While the number of court cases were particularly high in 2016 due to a special mobility procedure that caused major discontent, in the same speech the Minister acknowledges that there have been on average 1,000-1,200 Court cases encountered in previous years as well, and concludes that "there is probably a more structural issue". The shorthand document of the audition can be found on the website of the Italian Parliament: documenti.camera.it.

in teachers' ROLs are resolved based on the official school ordering. However, the DA-STB mechanism can be significantly inefficient. The intuition is that an applicant who ranks a school as part of a coarser region, could potentially secure another vacancy within the same region without adversely impacting their welfare while enhancing the assignments for certain teachers with more refined preferences. This reasoning underlies the design of the existing assignment system.

Nevertheless, this modification to the DA algorithm introduces the possibility of justified envy. In essence, a teacher whose acceptance to a region is deferred might eventually not be admitted to the region due to increased competition in later steps of the algorithm. Moreover, those teachers whose acceptances are being delayed because they rank a school as part of a coarse region, can be better off by strategically ranking the school instead of the region, implying that the current assignment mechanism is not strategy-proof either.[10]

We provide a practical solution to this problem. We introduce a new assignment mechanism called *Deferred Acceptance with Hierarchical Choice (DA-HC)*. A distinguishing feature of the DA-HC mechanism is that teacher applications and institutional choices (acceptances/rejections) happen at the regional level rather than at the school level. For example, if a teacher ranks a municipality as her top choice, in the first step, the teacher applies to the municipality, requesting a position in any of the municipality's schools. At each step, each province considers all applications to its schools and regions, and decides whether the applicants are tentatively accepted to its schools or rejected. The key innovative idea is that, instead of rejecting all the applicants who rank the school as part of a coarser region, we introduce a farsighted concept to determine which applicants can be safely rejected.

Given a set of applications to a province's schools and regions, we consider the maximum size (the number of admitted teachers) that can be achieved with those applications. If the maximum achievable size decreases when we remove a teacher's application, we call that teacher a *critical teacher*. At the beginning of each step of the DA-HC mechanism, we identify

[10]There is anecdotal evidence for such strategic behavior. The following question was posted by a teacher on an online forum accessible at orizzontescuolaforum.net (at that year, teachers could rank up to 20 items): "*In view of the coming mobility procedure, I would like to apply to come back home, to Catania. In order to come back to my family, I would be willing to go to any municipality in the province of Catania, and so I am thinking of indicating the province as a whole as the twentieth option in my rank order list. However, I was told that the more specific the requested item, the more chances you have of obtaining the transfer (in the sense that, if I have not misunderstood, if someone indicates a particular municipality or school, even if they have a lower score, they are prioritized over those who have generically indicated the entire province). Thus, how can I know which municipalities or, better, which schools in the province have free places for transfers, so that I can indicate them in the list?*"

critical teachers for each province, and it is those critical teachers who are asked to wait even if they have higher priority, when they are competing against finer applications. The DA-HC mechanism not only eliminates justified envy, but it is also strategy-proof and Pareto improves over the DA-STB mechanism.

A natural question is whether the DA-HC mechanism is optimal in efficiency terms subject to eliminating justified envy (JE). To address this question, we first show that two commonly used efficiency notions from the literature, namely *Pareto efficiency subject to eliminating justified envy* and *size efficiency subject to eliminating justified envy*, do not apply effectively in this context since they are not compatible with strategy-proofness. Motivated by these impossibilities, we introduce the following efficiency concept. An assignment is *Pareto-size efficient subject to eliminating justified envy* if it eliminates JE and there is no other assignment that also eliminates JE, Pareto dominates it, and, at the same time, assigns more teachers to acceptable schools. We show that the DA-HC mechanism is Pareto-size efficient subject to eliminating justified envy. This new efficiency concept plausibly formalizes the policymaker's objective as revealed in the design of the current mechanism.[11]

Using administrative data from Italian teacher assignment, we provide evidence that teachers indeed rank regions frequently and the DA-HC mechanism can potentially bring significant welfare improvements over the DA-STB mechanism in practice. In particular, comparing DA-HC and DA-STB outcomes on the subsample of preschool teachers, we show that DA-HC improves the assignment for 3.87% of the teachers. Our simulations show that DA-HC can potentially bring welfare improvement over the benchmark DA-STB mechanism for more than 10% of the teachers.

Organization of the paper. The rest of the paper is organized as follows. In Section 2, we briefly discuss the related literature. In Section 3, we present our teacher assignment problem and discuss the desiderata. In Section 4, we discuss the current mechanism. In Section 5 we introduce the DA-HC mechanism and discuss its properties. In Section 6, we provide evidence for the potential improvements in teacher welfare from implementing DA-HC as opposed to the benchmark alternative DA-STB. Section 7 concludes. All proofs are in the Appendix A. Additional details on the institutional context are in the Appendix B.1.

[11]As such, our study also contributes to the minimalist market design (Sönmez, 2023) literature by formalizing the primary objectives of an institution, identifying the root cause of the problem, and proposing a solution with minimal interference. See Greenberg et al. (2024) and Sönmez and Ünver (2022) for other recent examples.

2 Related Literature

This paper contributes to the literature on matching and market design. While there are studies on decentralized teacher assignment systems (Bates et al., 2022; Biasi, 2021; Biasi et al., 2021), there is also a growing literature studying centralized teacher assignment systems around the world such as France (Combe et al., 2022a,b), Mexico (Pereyra, 2013), Peru (Bobba et al., 2021; Ederer, 2023), Turkey (Dur and Kesten, 2019), and Teach for America in Chicago (Davis, 2022).

In a recent study, Combe et al. (2022b) propose a new teacher assignment mechanism for France and show that it improves over the benchmark DA mechanism in that context. Different from theirs, in our context teacher ROLs include ties while school priority orderings are strict. In fact, one of our contributions to the literature is to introduce a novel teacher assignment model which incorporates indifferences in preferences structured around a geographical hierarchy.

Our main theoretical and conceptual contributions belong to the literature on priority-based matching with indifferences, e.g., Abdulkadiroğlu et al. (2009), Erdil and Ergin (2008, 2017), Erdil and Kumano (2019), Irving (1994), Irving and Manlove (2008), and Manlove (2002). This literature has typically focused on indifferences arising in priorities in contrast to preferences (Abdulkadiroğlu et al., 2009; Erdil and Ergin, 2008). Erdil and Ergin (2017) allow indifferences both in preferences and priorities. Our results, in particular our impossibility results, are independent from their results as it is crucial that indifferences are allowed only in preferences in our context.

Jaramillo and Manjunath (2012) and Alcalde-Unzu and Molis (2011) consider an object allocation problem where agents' preferences may include indifferences and agents are endowed with an object, like in our setting. However, unlike our setting, there is no given priority structure to be respected. In fact, their main contribution is to introduce a class of mechanisms that are strategy-proof, Pareto-efficient, and individually rational, while Pareto efficiency is incompatible with the central policy objectives in our context.

Manlove et al. (2002) show that finding a matching that is size efficient subject to eliminating justified envy is NP-hard in the presence of indifferences, and their result implies that in our context, finding a matching that is size efficient subject to eliminating JE is NP-hard. We contribute to this literature by establishing that imposing strategy-proofness turns the already computationally hard problem into an impossibility. Moreover, we introduce a novel efficiency concept which weakens size efficiency subject to eliminating justified envy, and show that the problem becomes polynomial-time solvable with a strategy-proof algorithm when indifferences are structured around a hierarchy.

Finally, there are earlier studies in different contexts showing that hierarchical structures can be intimately related with achieving certain other objectives. However, a hierarchical structure is imposed on different objects than those in our context, such as on the quota structure in Biró et al. (2010), on the random assignment constraint structure in Budish et al. (2013), and on the distributional constraint structure in Kamada and Kojima (2018).

3 Model

We consider the problem of reassigning teaching positions in schools among tenured teachers. Let $T = \{t_1, \ldots, t_{|T|}\}$ be a finite set of teachers and $S = \{s_1, \ldots, s_{|S|}\}$ be a finite set of schools. Let $\mathcal{G}$ be a finite set of (geographical) regions where each region consists of a set of schools, that is, for each $G \in \mathcal{G}$, $\emptyset \neq G \subseteq S$ and $\cup_{G \in \mathcal{G}} G = S$. We assume that for any pair of regions $G, G' \in \mathcal{G}$, the sets of schools in the two regions are either distinct ($G \cap G' = \emptyset$) or nested ($G \subseteq G'$ or $G' \subseteq G$). This assumption captures real-world geographical structures. In Italy, the regions are municipalities, districts, and provinces, where each school belongs to a unique municipality, each municipality belongs to a unique district, and each district belongs to a unique province.[12]

Each teacher $t \in T$ is initially assigned to a school, which we call her **endowment school** and denote by $\omega_t \in S$. Each school $s \in S$ has a capacity $q_s \in \mathbb{N}$ and a (strict) priority ordering $\succ_s$ over the teachers, which is a complete, transitive, and anti-symmetric binary relation over T.[13] Let $q = (q_s)_{s \in S}$ be the capacity profile. We assume $\sum_{s \in S} q_s \geq |T|$.[14]

We assume that for each teacher $t \in T$ and her endowment school ω_t, t is one of the top

[12]While this is a precise description of geographical regions as far as teacher assignment is concerned, some Italian metropolitan cities that are big municipalities include sub-regions that are sometimes referred to as districts as well. See Appendix B.1 for details on the geographical hierarchy.

[13]Note that, our model also captures assignment problems where no school places are initially owned by any teacher but teachers have outside options, since we can include a null school $s_{null} \in S$ with capacity $q_{s_{null}} \geq |T|$ and let $\omega_t = s_{null}$ for each $t \in T$.

[14]In practice, the capacity of each school is essentially determined by the number of teachers who are initially assigned to that school and who participate in the reassignment system, and possibly some newly created vacancies. Therefore, the total number of school places is greater than the total number of teachers. See Appendix B.3 for information about all sources of vacancies.

q_{ω_t} teachers in $\succ_{\omega_t}$.[15]

Teachers' Preferences

In the Italian teacher assignment, each teacher communicates her preferences by submitting a rank order list (ROL) of items where each item is either a school or a region. For example, a teacher t might rank a school s as her first choice, a municipality G such that $s \in G$ as her second choice, and another school $s' \notin G$ as her third choice. The interpretation is that t (strictly) prefers s to any other school in G and also to s', and t is indifferent between all schools in $G \setminus \{s\}$ and prefers each of them to s'. In general, how a teacher compares any two schools is determined by the highest-rank occurrences of these two schools in the teacher's ROL, and this constitutes the basis for verifying priority violations and evaluating efficiency.

Accordingly, we assume that each teacher $t \in T$ has a preference relation represented by an ROL R_t that ranks schools and regions from the most preferred to the least preferred. That is, R_t is a transitive and anti-symmetric binary relation over $S \cup \mathcal{G}$, where the highest-ranked occurrences of any pair of schools determine preferences over these two schools.

Formally, given any school $s \in S$ and an ROL R_t, let $rank(s, R_t)$ denote the *highest-rank occurrence* of school s in R_t. That is, the $rank(s, R_t)$-st highest-ranked item in R_t is either s or a region that includes s, and there is no higher-ranked item in R_t that is either s or a region that includes s.[16]

Given any pair of schools $s, s' \in S$, teacher t **weakly prefers** s to s', denoted by $s \; \overline{R}_t \; s'$, if $rank(s, R_t) \leq rank(s', R_t)$; and (strictly) **prefers** s to s', denoted by $s \; \overline{P}_t \; s'$, if $rank(s, R_t) < rank(s', R_t)$. If $rank(s, R_t) = rank(s', R_t)$, t is **indifferent** between s and s', denoted by $s \; \overline{I}_t \; s'$.

[15] In Italy's teacher assignment, school priorities are determined by a teacher point score, in addition to other factors such as the geographical location of teachers' endowment schools. In particular, each teacher is in the top tier of her endowment school's priority ordering. School priorities also account for the geographical priorities, such that teachers have higher priority at the schools in their current municipality over teachers from other municipalities, and similarly they have higher priority at the schools in their current province over teachers from other provinces. Appendix B.1 provides a detailed description of how school priorities are determined.

[16] For instance, if $rank(s, R_t) = 1$, either s or a region that includes s is top-ranked in R_t.

Matchings and Mechanisms

A (teacher assignment) problem is a tuple $(T, S, \mathcal{G}, \omega, q, R, \succ)$. When the rest of the problem in question is clear, we simply denote a problem by the ROL profile R.

A matching is an assignment of teachers to schools in a way that respects capacity constraints. Formally, a **matching** is a correspondence $\mu : T \cup S \to T \cup S$, such that for each $t \in T$ and each $s \in S$, (i) $\mu(t) \in S$, (ii) $\mu(s) \subseteq T$, (iii) $\mu(t) = s$ if and only if $t \in \mu(s)$, and (iv) $|\mu(s)| \leq q_s$.

A **mechanism** elicits ROLs from the teachers and produces a matching. Given a mechanism φ and an ROL profile R, we denote the assignment of $t \in T$ by $\varphi_t(R)$.

Design Objectives

Individual Rationality. A natural design objective is that no teacher's assignment is worse than her endowment school. Formally, given a problem R, a matching μ satisfies **individual rationality** if for each $t \in T$, $\mu(t)\ \overline{R}_t\ \omega(t)$.

Fairness. A central design objective in Italian teacher assignment is that the matching respects school priorities by eliminating justified envy (JE). Given a problem R, a matching μ **eliminates JE** if whenever a teacher t envies the assignment of another teacher t', t' has a higher priority at her assigned school than t. That is, if $\mu(t')\ \overline{P}_t\ \mu(t)$, then $t' \succ_{\mu(t')} t$.

Incentives. Another important objective is to make it safe for the teachers to report their preferences truthfully. A mechanism φ is **strategy-proof** for teachers if, given a true ROL profile $(R_t)_{t \in T}$, no teacher $t \in T$ can benefit by misreporting her ROL. That is, for any other ROL R'_t of t, we have $\varphi_t(R)\ \overline{R}_t\ \varphi_t(R'_t, R_{-t})$.

In Italy, providing incentives for the teachers to report their preferences truthfully is an important policy concern and teachers are often advised to report truthfully, as suggested by the following quotation from a website specialized in education:[17]

> "The system proceeds by examining the preferences in the order indicated (from first to last). When the teacher is satisfied in one of the preferences the system

[17] Accessible at dimascuola.blogspot.com.

does not go further. At this point it is advisable to indicate the preferences simply according to your preferred order."

However, various features of the system such as the constraint on the length of the ROL make it difficult to ensure a strong form of incentive compatibility, since the teachers are facing a non-trivial "portfolio choice problem" as well.[18] On the other hand, our strategy-proofness concept ensures incentive compatibility to the extent possible in this setting since it incentivizes truthful ranking of the schools and regions in the final application portfolio.

Efficiency. An important efficiency requirement is that no teacher should prefer an unassigned seat to her assignment. Formally, given a problem R, a matching μ is **non-wasteful** if there is no $t \in T$ and $s \in S$ such that $s\ \overline{P}_t\ \mu(t)$ and $|\{t' \in T : \mu(t') = s\}| < q_s$.

A natural measure of efficiency is based on the Pareto dominance relation with respect to teachers' preferences. A matching μ Pareto dominates another matching μ' if every teacher weakly prefers μ to μ', and at least one teacher (strictly) prefers μ to μ'. Given a problem R, a matching μ is **Pareto efficient** if there is no matching μ' that Pareto dominates μ. A matching is **Pareto efficient subject to eliminating JE** if it eliminates JE and it is not Pareto dominated by any other matching that also eliminates JE. Note that Pareto efficiency subject to eliminating JE implies no-wastefulness.

Another natural measure of efficiency is the number of teachers who move to a better school than their endowment school. Given a problem R, a matching μ size dominates another matching μ' if μ assigns more teachers to acceptable schools than μ', i.e., $|\{t \in T : \mu(t) \neq \omega_t\}| > |\{t \in T : \mu'(t) \neq \omega_t\}|$. A matching μ is **size efficient** if there is no matching μ' that size dominates it. A matching μ is **size efficient subject to eliminating JE** if μ eliminates JE and there is no matching μ' that also eliminates JE while assigning more teachers to acceptable schools. Note that size efficiency subject to eliminating JE does not imply no-wastefulness.

We show that both *Pareto efficiency subject to eliminating JE* and *size efficiency subject to eliminating JE* are incompatible with *strategy-proofness*.[19]

[18] See for example Ali and Shorrer (2021), Calsamiglia et al. (2010), Chade and Smith (2006), and Haeringer and Klijn (2009).

[19] Erdil and Ergin (2017) similarly show that there is no mechanism that is strategy-proof and Pareto efficient subject to eliminating justified envy when indifferences are allowed on both sides. However, their result crucially relies on having indifferences in priorities and therefore does not imply our impossibility result.

Proposition 1. *There is no mechanism that is strategy-proof and Pareto efficient subject to eliminating JE.*

Proposition 2. *There is no mechanism that is strategy-proof, non-wasteful, and size efficient subject to eliminating JE.*

Therefore, we introduce the following efficiency notion which is weaker than both. A matching μ is **Pareto-size efficient subject to eliminating JE** if it eliminates JE and there is no other matching μ' that also eliminates JE, Pareto dominates μ and, at the same time, assigns more teachers to acceptable schools than μ. Note that Pareto-size efficiency subject to eliminating JE does not imply no-wastefulness.

4 The Current Mechanism: Deferred Acceptance with Hierarchical Priorities

After receiving ROLs from the teachers, the Italian Ministry of Education determines the matching outcome by running a version of the deferred acceptance algorithm (Gale and Shapley, 1962) with the following critical feature.[20]

In the course of the assignment algorithm, if the next item to be considered in a teacher's ROL is a region (as opposed to a single school), the algorithm considers the teacher for the schools in that region one by one following a pre-determined official ordering of the schools.[21] We assume that the indexing of the schools in our model is consistent with the official ordering. Moreover, teachers who rank a school as part of a finer region are given priority over teachers ranking the same school as part of a coarser region, provided that the coarser region has at least one school with vacancy at that step of the algorithm. Intuitively, the policymaker's intention is that teachers who rank coarser regions could be assigned another school in the same indifference class later.[22]

We call this algorithm "Deferred Acceptance with Hierarchical Priorities" where "Hierarchical Priorities" refers to the critical feature that the geographical hierarchy among ranked

[20] Fundamental sources that explain the principles underlying the assignment mechanism are the national collective bargaining agreement (*Contratto Collettivo Nazionale Integrativo*) and the ministerial decree about teacher mobility (*Ordinanza sulla mobilità personale docente, educativo ed ATA*).

[21] This ordering is called Official List (*Bollettini Ufficiali*) and it is published each year on the website of the Ministry of Education (www.istruzione.it).

[22] This feature is reported in art. 6, par. 5 of the national collective bargaining agreement (*Contratto Collettivo Nazionale Integrativo*).

items induce additional artificial priority rights. A formal definition follows.

Deferred Acceptance with Hierarchical Priorities (DA-HP)

At any step of the algorithm, we say that a school has *vacancy* if the total number of applications to the school until and including that step is less than the school's capacity.

Step 1: Each teacher applies to the smallest-index school in their top-ranked item (a school or a region) in their ROL. Each school s first considers the applicants to s whose top-ranked item does not include a school with some vacancy.[23] Among those, the highest priority applicants are tentatively accepted until there is no applicant or available seat left. If there are available seats left, among the remaining applicants, the highest priority applicants are tentatively accepted until there is no applicant or available seat left. Teachers who are not tentatively accepted to any school are rejected. If there is no rejection at this step, then stop and return the resulting matching. Otherwise go to Step 2.

Step $k \geq 2$: Each teacher applies to the smallest-index school in their highest-ranked item (a school or a region) in their ROL that includes a school that has not rejected them before (if there is no such item, the teacher applies to her endowment school). Each school s first considers the applicants to s whose highest-ranked item in their ROL that includes a school that has not rejected them before, does not include a school with some vacancy. Among those, the highest priority applicants are tentatively accepted until there is no applicant or available seat left. If there are available seats left, among the remaining applicants, the highest priority applicants are tentatively accepted until there is no applicant or available seat left. Teachers who are not tentatively accepted to any school are rejected. If there is no rejection at this step, then stop and return the resulting matching. Otherwise go to Step $k + 1$.

The algorithm must eventually stop because no teacher applies twice to any school and teachers will never be rejected by their endowment schools. The following example illustrates

[23] Note that if a teacher t is considered while another applicant t' is not, it must be that t has a finer application than t'.

the workings of DA-HP and shows its two important shortcomings: DA-HP does not eliminate JE and it is not strategy-proof.

Example 1. Let $T = \{t_1, t_2, t_3\}$, $S = \{s_1, s_2, s_3, s_4\}$, and $\mathcal{G} = \{r_1, r_2, \phi = \{r_1, r_2\}\}$ with $r_1 = \{s_1, s_2\}$ and $r_2 = \{s_3, s_4\}$. That is, ϕ is the highest level region, which has two distinct subregions r_1 and r_2. Let $q_{s_1} = q_{s_2} = q_{s_3} = 1$, $q_{s_4} = 4$, $\omega_t = s_4$ for each $t \in T$, and ROLs and priorities be as depicted below.

R_{t_1}	R_{t_2}	R_{t_3}	R_{t_4}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
r_1	s_1	s_3	s_3	t_1	t_3	t_4	t_1
		s_2		t_3	t_2	t_3	t_2
				t_2	t_1	t_1	t_3
				t_4	t_4	t_2	t_4

In the first step of DA-HP, t_1 and t_2 apply to s_1, and t_3 and t_4 apply to s_3. At this step, although $t_1 \succ_{s_1} t_2$, t_1 is rejected and t_2 is tentatively accepted by s_1 since s_2, another school in t_1's ranked region r_1, has vacancy at this step (note that s_2 has not received any applications so far). Later in the second step, t_1 is rejected by s_2 as well since t_3 applies to s_2 in the second step and $t_3 \succ_{s_2} t_1$.

The DA-HP outcome for this example does not eliminate JE because t_1 misses out at s_1 at the expense of a lower priority teacher t_2. Moreover, the DA-HP mechanism is not strategy-proof since t_1 can guarantee her more preferred school s_1 by manipulating her ROL, for example, by reporting only s_1 as acceptable.

5 Deferred Acceptance with Hierarchical Choice

In our proposed design, teacher applications and institutional choices (acceptances/rejections) happen at the regional level rather than at the school level. For example, if a teacher ranks a municipality as her top choice, in the first step, the teacher applies to the municipality, requesting a position in any of the municipality's schools. At each step, provinces (the highest level regions) consider all applications to their schools and regions, and decide whether the applicants are tentatively accepted to their schools or rejected.

For this purpose, we first design a choice rule for each province, called the Hierarchical Choice Rule. This choice rule determines, for each set of applications to the province, which applications are accepted and to which schools.[24] Afterwards, we introduce the deferred acceptance mechanism with hierarchical choice, which uses the hierachical choice rules that we have designed in an otherwise standard deferred acceptance algorithm.

Hierarchical choice rule

Let Φ be the set of largest regions in $\mathcal{G}$, i.e., the set of regions that are not contained in any other region. In the Italian teacher assignment, provinces are the largest regions. We will be referring to the largest regions simply as provinces from now on.

Consider any province, say $\phi \in \Phi$. Let S^ϕ be the set of schools in ϕ and G^ϕ be the set of regions contained in ϕ (e.g., districts and municipalities of the province ϕ in the Italian teacher assignment). An **application** to ϕ is a pair $(t, x) \in T \times (S^\phi \cup G^\phi \cup \{\phi\})$. Note that an application may be an application to a single school in the province, to a region contained in the province, or to the province as a whole. We call a set of applications A **plausible** if each teacher has at most one application (which might include a single school or a region). Let $\mathcal{A}^\phi$ denote the set of all plausible sets of applications to ϕ.

Let $A \in \mathcal{A}^\phi$. For each $t \in T$ with application $(t, x) \in A$, we denote the set of **feasible schools** for t by $F(x)$, where $F(x) = \{x\}$ if x is a single school and $F(x) = x$ if x is a region. A province-level choice rule describes which applications are accepted and to which schools, from any possible plausible set of applications. Formally, a **province-level choice rule** is a function C^ϕ that associates each plausible set of applications $A \in \mathcal{A}^\phi$, with an assignment of teachers to schools such that (i) for each $t \in T$ with application $(t, x) \in A$, the assignment of t is either a feasible school or nothing, denoted by $C_t^\phi(A) \in F(x) \cup \{\emptyset\}$, and (2) for each $s \in S^\phi$, no more than q_s teachers are assigned to s.

We say teacher t with application $(t, x) \in A$ has a **finest application** if there is no $(t', x') \in A$ such that $F(x') \subsetneq F(x)$. We say teacher t with application $(t, x) \in A$ is the **smallest-index teacher among finest applications** if t has the smallest index among teachers who have a finest application.

Let $A \in \mathcal{A}^\phi$ be given. An applicant t with application $(t, x) \in A$ is **critical (for maximizing**

[24] In fact, we do not define the choice rule for the entire domain of sets of applications where a set of applications might include multiple applications from the same teacher. Instead, we define it only for "plausible" sets of applications that include at most one application (which might be to a region) from each teacher. This is sufficient since in the deferred acceptance mechanism that uses these choice rules, a province never faces multiple applications from the same teacher. See also Doğan and Erdil (2022) who enable their entire design based on this insight.

the size) in A if the maximum size matching achievable with A (by assigning them to their feasible schools while respecting the capacity profile q) has greater size than the maximum size matching achievable with $A \setminus \{(t,x)\}$. Observe that an applicant is critical in A if and only if the applicant is assigned to a school in every maximum size matching achievable with A. The following example illustrates the concept of a critical applicant.

Example 2. Let us reconsider the problem in Example 1. Consider $A = \{(t_1, r_1), (t_2, s_1), (t_3, s_3), (t_4, s_3)\} \in \mathcal{A}^\phi$. Note that the size of a maximum size matching in A is 3 (the total number of seats in the feasible schools is three, which can all be allocated by, for example, assigning t_1 to s_2, t_2 to s_1, and t_3 to s_3).

The size of the maximum size matching in $A \setminus \{(t_1, r_1)\}$ is 2. Since $3 > 2$, t_1 is critical in A. The size of the maximum size matching in $A \setminus \{(t_2, s_1)\}$ is 2. Since $3 > 2$, t_2 is critical in A. The size of the maximum size matching in $A \setminus \{(t_3, s_3)\}$ is 3. Therefore, t_3 is not critical in A. Similarly, t_4 is also not critical in A. Hence, in A, only teachers t_1 and t_2 are critical.

Next, we introduce the *Hierarchical Choice Rule*, which considers applications starting from the smallest-index teacher among finest applications. Moreover, whenever a teacher is considered, if they are identified as critical, then they are assigned to a vacant seat instead of replacing lower priority teachers in other schools (which can be interpreted as being asked to wait). Formally, given $A \in \mathcal{A}^\phi$, the Hierarchical Choice $HC^\phi(A)$ is determined via the following algorithm.

Hierarchical Choice Rule (HC^ϕ)

Step 1**:** Consider the smallest-index teacher among finest applications, say (t,x). Tentatively accept t to the school in $F(x)$ with the smallest index. Move to the next step.

Step $k \geq 2$**:** Among the teachers who are not tentatively accepted by any school and who have not been rejected by all their feasible schools (if there is none, terminate), consider the smallest-index teacher among finest applications, say (t,x).

Case 1: If t is critical in A, tentatively accept t to the smallest index school in $F(x)$ that has a vacant seat.[25]

[25] By Lemma 3, there exists a school in $F(x)$ that has a vacant seat at this step.

Case 2: If t is not critical in A, consider the smallest index school $s \in F(x)$ that has not rejected t before. Tentatively accept t to s if either s has a vacant seat or s does not have a vacant seat but t has a higher priority than the lowest-priority tentatively accepted teacher, say t' (by rejecting t' from s if it is the latter case). Otherwise, reject t from s.

In plain words, $HC^{\phi}(A)$ is based on a deferred acceptance type algorithm (although it does not take any preference information as input) where teachers with applications in A apply to their feasible schools following the official school ordering (or just the indices). The important features are that: (1) teachers apply one by one and the teachers with finer applications have precedence in the application order and (2) if the next applicant is a critical teacher, she is tentatively accepted to her smallest index feasible school among her feasible schools with vacancy (and we know by Lemma 3 that such a school exists). If she is not a critical teacher, then we proceed as in a standard deferred acceptance procedure (the teacher possibly replacing a lower priority teacher who was tentatively accepted in an earlier step). The following example illustrates the workings of HC^{ϕ}.

Example 3. Let us reconsider the problem in Example 1. Consider $A = \{(t_1, r_1), (t_2, s_1), (t_3, s_3), (t_4, s_3)\} \in \mathcal{A}^{\phi}$. In Step 1, consider the smallest-index teacher among finest applications, which is (t_2, s_1). Tentatively accept t_2 to the school in $F(s_1)$ with the smallest index, which is s_1.

In Step 2, among the teachers who are not tentatively accepted by any school and who have not been rejected by all their feasible schools, consider the smallest-index teacher among finest applications, which is (t_3, s_3). Since t_3 is not critical in A as we have shown in Example 2, consider the smallest index school $s \in F(s_3)$ that has not rejected t_3 before, which is s_3. Tentatively accept t_3 to s_3 since s_3 has a vacant seat.

In Step 3, among the teachers who are not tentatively accepted by any school and who have not been rejected by all their feasible schools, consider the smallest-index teacher among finest applications, which is (t_4, s_3). Since t_4 is not critical in A as we have shown in Example 2, consider the smallest index school $s \in F(s_3)$ that has not rejected t_4 before, which is s_3. Since $t_4 \succ_{s_3} t_3$, tentatively accept t_4 to s_3 and reject s_3.

In Step 4, among the teachers who are not tentatively accepted by any school and who have not been rejected by all their feasible schools, consider the smallest-index teacher among finest applications, which is (t_1, r_1). Since t_1 is critical in A as we have shown in Example 2, tentatively accept t_1 to the smallest index school in $F(r_1)$ that has a vacant seat, which is s_2. Hence, $HC^{\phi}(A) = \{(t_1, s_2), (t_2, s_1), (t_4, s_3)\}$.

DA mechanism with hierarchical choice

Next, we introduce a matching mechanism based on a deferred acceptance algorithm (Gale and Shapley, 1962) where each province ϕ considers applications to its schools and regions according to the HC^{ϕ} rule.

Deferred Acceptance with Hierarchical Choice (DA-HC)

Step 1: Each teacher applies to their top-ranked item (a school or a region) in their ROL. Each province ϕ considers applications (from this step) to its schools or regions and tentatively accepts applicants to its schools via HC^{ϕ}. Applicants who are not tentatively accepted to any school are rejected. If there is no rejection by any province at this step, then stop and return the resulting matching. Otherwise go to Step 2.

Step $k \geq 2$: Teachers who were rejected in the previous step apply to their next-best acceptable item in their ROL (If there is no such item, they apply to their endowment schools). Each province ϕ considers its tentatively accepted applicants from the previous step together with its applicants from this step, and tentatively accepts applicants to its schools via HC^{ϕ}. Applicants who are not tentatively accepted to any school are rejected. If there is no rejection by any province at this step, then stop and return the resulting matching. Otherwise go to Step $k + 1$.

The algorithm must eventually stop because no teacher applies twice to any item in their ROL and teachers will never be rejected by their endowment schools. The following example illustrates the workings of the DA-HC mechanism.

Example 4. Let us reconsider the problem in Example 1. In Step 1, each teacher applies to their top-ranked item (a school or a region) in their ROL and the province ϕ considers applications (from this step) to its schools or regions, which is $A = \{(t_1, r_1), (t_2, s_1),$

$(t_3, s_3), (t_4, s_3)\}$, and tentatively accepts applicants to its schools via HC^ϕ, which is $HC^\phi(A) = \{(t_1, s_2), (t_2, s_1), (t_4, s_3)\}$ as we have shown in Example 4.

In Step 2, teachers who were rejected in the previous step, which is t_3, apply to their next-best acceptable item in their ROL, which is s_2. The province ϕ considers its tentatively accepted applicants from the previous step together with its applicants from this step, which is $A' = \{(t_1, r_1), (t_2, s_1), (t_3, s_2), (t_4, s_3)\}$. Observe that only t_4 is critical in A' and $HC^\phi(A') = \{(t_1, s_1), (t_3, s_2), (t_4, s_3)\}$, according to which ϕ tentatively accepts applicants to its schools.

In Step 3, t_2 applies to her endowment school s_4 and gets accepted, and the algorithm stops. The final DA-HC outcome is $\{(t_1, s_1), (t_2, s_4), (t_3, s_2), (t_4, s_3)\}$.

Theorem 1. *The DA-HC mechanism is individually rational, strategy-proof, non-wasteful, and Pareto-size efficient subject to eliminating JE.*

A complete proof of Theorem 1 is in Appendix A. Here, we only highlight the key ideas the proof builds on. The individual rationality of the DA-HC mechanism directly follows from the fact that a teacher is never rejected by her endowment school.

Some key observations about HC^ϕ are crucial for establishing the desirable properties of the DA-HC mechanism. For any Step k of the $HC^\phi(A)$ algorithm, let (A^k, q^k) denote the reduced problem at the beginning of Step k, i.e., A^k is the set of applications from teachers who are not tentatively accepted by any school and who have not been rejected by all their feasible schools by the beginning of Step k, and q^k is the profile of vacant seats at the beginning of Step k. Take any teacher t with some $(t, x) \in A^k$. We show in Lemma 3 that t is critical in A if and only if t is critical in A^k when the capacity profile is q^k. This allows us to identify the set of critical teachers only for the entire problem in the definition of HC^ϕ, instead of identifying critical teachers for each reduced problem at every step. Moreover, this property implies that whenever a critical teacher is considered at any step, there must be a vacant seat at some feasible school for the teacher. We show in Lemma 5 that, if at some step of the $HC^\phi(A)$ algorithm, a teacher who is critical in A is assigned to a school s, then no teacher is ever rejected, in particular t is never rejected, by s in the course of the $HC^\phi(A)$ algorithm. This property underlies the farsightedness in the notion of a critical teacher. In particular, it is guaranteed that the critical teachers will not be displaced from their assigned schools in the later steps of HC^ϕ algorithm.

In order to show that the DA-HC mechanism is strategy-proof and eliminates JE, building on the important properties HC^{ϕ} that we have established, we show that HC^{ϕ} satisfies the following key properties:

1. HC^{ϕ} always chooses a maximum size matching (Proposition 3).

2. HC^{ϕ} always eliminates JE (Proposition 4).

3. HC^{ϕ} satisfies substitutability in the sense that, if a teacher is accepted to a school from a set of applications, the teacher will still be accepted to a school (not necessarily the same school) if any application from another student is removed from the set of applications (Proposition5).

The rest of the proof associates the problem with an auxiliary matching with contracts problem (Hatfield and Milgrom, 2005) and invokes concepts and results from that literature (Hatfield and Kominers, 2019).

DA with simple tie-breaking

When eliminating JE and strategy-proofness are of concern, the natural benchmark is the DA mechanism with simple tie-breaking (DA-STB) where ties in teachers' ROLs are resolved based on the official school ordering.

Deferred Acceptance with Simple Tie-Breaking (DA-STB)

Step 1: Each teacher applies to the smallest-index school in their top-ranked item (a school or a region) in their ROL. Each school s considers applications (from this step). Among those, the highest priority applicants are tentatively accepted until there is no applicant or available seat left. Teachers who are not tentatively accepted to any school are rejected. If there is no rejection by any school at this step, then stop and return the resulting matching. Otherwise go to Step 2.

Step $k \geq 2$: For each teacher t who was rejected in the previous step, consider the best item in her ROL that includes a school that has not rejected t before (if there is no such item, the teacher applies to her endowment school). Among the schools in that item, teacher t applies to the smallest-index school that has not rejected her before.

Each school s considers its tentatively accepted applicants from the previous step together with its applicants from this step. Among those, the highest priority applicants are tentatively accepted until there is no applicant or available seat left. Teachers who are not tentatively accepted to any school are rejected. If there is no rejection by any school at this step, then stop and return the resulting matching. Otherwise go to Step $k+1$.

The DA-STB mechanism is also strategy-proof and eliminates justified envy (Abdulkadiroğlu and Sönmez, 2003). However, we show that the DA-HC mechanism both Pareto dominates and size dominates DA-STB.[26]

Theorem 2. *The DA-HC mechanism both Pareto dominates and size dominates the DA-STB mechanism.*

A complete proof of Theorem 2 is in Appendix A. Here, we provide an example of a problem where a teacher prefers her DA-HC assignment to her DA-STB assignment and more teachers are assigned to their acceptable positions in the DA-HC assignment.

Example 5. Let $T = \{t_1, t_2\}$, $S = \{s_1, s_2, s_3\}$, and $\mathcal{G} = \{\phi = \{r_1, r_2\}\}$ with $r_1 = \{s_1, s_2\}$ and $r_2 = \{s_3\}$. That is, there is only one province, which has two subregions. Let $q_{s_1} = q_{s_2} = 1$, $q_{s_3} = 2$, $\omega_t = s_3$ for each $t \in T$, and ROLs and priorities be as depicted below.

R_{t_1}	R_{t_2}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
r_1	s_1	t_1	t_1	t_1
		t_2	t_2	t_2

Note that in the DA-STB assignment, t_1 is assigned to s_1 and t_2 remains in her endowment school s_3, while in the DA-HC assignment, t_1 is assigned to s_2 and t_2 is assigned to s_1. Hence, the DA-HC assignment both Pareto improves and size improves over the DA-STB assignment by making t_2 better off while leaving t_1 indifferent, and increasing the number of teachers assigned to their acceptable positions from 1 to 2.

[26]In contrast, in a model where school priorities include indifferences while teacher preferences are strict, there is no strategy-proof mechanism that Pareto improves over the DA-STB mechanism (Abdulkadiroğlu et al., 2009).

6 Evidence for Potential Improvements from DA-HC

Theorem 2 shows that the DA-HC mechanism is theoretically superior to the natural benchmark, the DA-STB mechanism. But in practice, should we expect significant gains from using the DA-HC mechanism as opposed to the DA-STB mechanism? Note that the significance of the improvement especially relies on the prevalence of ranked regions in teachers' ROLs, e.g., if teachers never rank regions but only rank singleton schools, then two mechanisms would be empirically equivalent.

To answer this question, we use administrative data from the Italian Ministry of Education on the universe of applications for the teacher reassignment procedure in the school-years 2019-20, 2020-21, and 2021-22. We provide evidence that teachers indeed rank regions frequently, and the DA-HC mechanism can potentially bring significant welfare improvements over the DA-STB mechanism in practice. Importantly, the data include the geographical hierarchy, schools' capacities, submitted teacher ROLs, teachers' scores and information about specific priority's eligibility.[27]

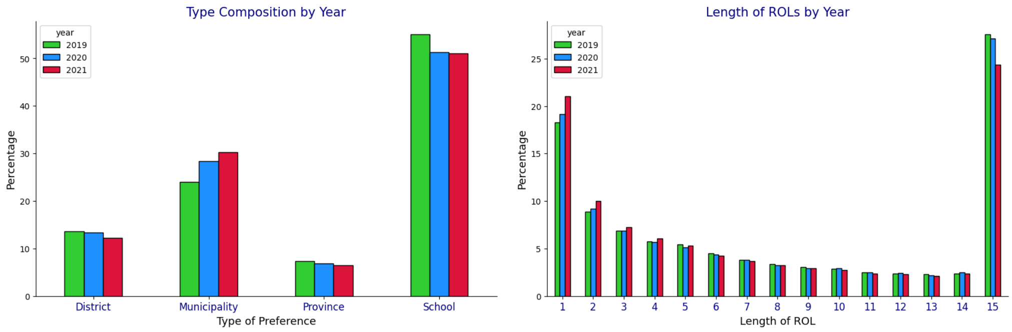
First, we report some descriptives showing the prevalence of regions in the ROLs based on all applications (there were around 100,000 applications each year). Teachers can submit an ROL of up to 15 items. Approximately 27% of the applications include 15 items (hence, 73% of the teacher applications do not use all available slots in their ROLs). On the other hand, around 20% of the applications report only one item (Figure 1). On average, the ROLs include 7-8 items. Overall, 50-55% of the ranked items are schools, 25-30% are municipalities, 10-12% are districts, and 6-7% are provinces (Figure 1).[28] Also, teachers do not necessarily rank regions in the lower part (possibly outcome irrelevant part) of their ROLs since, for example, 19-24% of the first ranked items, 27-34% of the second ranked items, and 32-38% of the third ranked items are regions (Figure 2). Moreover, the frequency of ranked regions is fairly consistent over the three school-years 2019-20, 2020-21, and 2021-22.

Second, we provide further evidence that DA-HC mechanism could bring significant

[27]We provide details on the data sources and additional descriptives in Section B.3. As we explain in the Appendix, the school priorities are based on teacher scores along with other criteria, from which we identify more than 40 tiers. Unfortunately, we did not have access to enough information to precisely identify some of those criteria. Therefore, we cannot provide the actual number of priority violations.

[28]Most of the teachers have family located in the southern regions (75% of those who ask to move for family reasons) while most of the vacancies are on the northern regions (Figure 3). Because of that, teachers typically have preferences for certain regions at some point in their career (Figure 4).

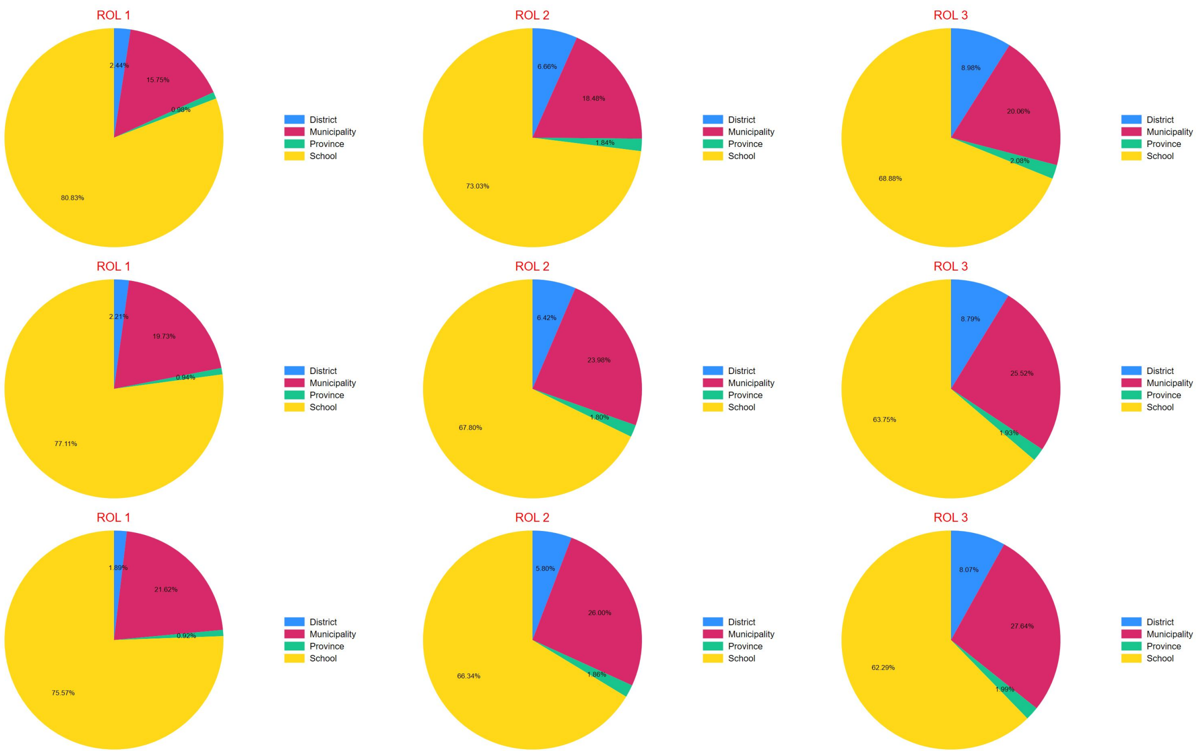
Figure 1: ROLs



(a) Type composition (b) Length

Notes: Figure (a) shows the percentage of schools and regions (municipalities, districts and provinces) among all items in the submitted ROLs. Figure (b) shows the percentage of ROLs of a given length.
Source: Italian Ministry of Education, Restricted Data.

Figure 2: ROL Type Composition by Rank



Notes: These pie charts show the distribution of single schools and regions among the top ranked items (ROL 1), second ranked items (ROL 2), and third ranked items (ROL3) in 2019-20 (first row), 2020-21 (second row), and 2021-22 (third row).
Source: Italian Ministry of Education, Restricted Data.

improvements as opposed to the DA-STB mechanism by running the two algorithms and comparing their outcomes. For simplicity, we run DA-STB and DA-HC algorithms on a large subsample of our data,[29] namely the subsample of preschool teachers, which constitute between 12-13% of all applications (Table 1).[30]

Table 1: Applications by School Type

School Type	*2019-20*	*2020-21*	*2021-22*
Preschool	12.48%	12.95%	12.76%
Primary School	27.77%	28.87%	29.34%
Middle School	18.90%	18.07%	17.13%
High School	40.85%	40.11%	40.78%
Total Applications	129,803	108,677	87,454

Notes: This table reports the fraction of applications per type of school. We report data for school years from 2019-20 to 2021-22.
Source: Italian Ministry of Education, Restricted Data.

Comparing DA-HC and DA-STB outcomes on this subsample, we observe that 3.87% of the teachers prefer DA-HC over DA-STB, which suggests a significant improvement from DA-HC over DA-STB. In line with our theoretical results, we find that no teacher prefers DA-STB over DA-HC.[31]

Note that DA-HC improves over the current mechanism DA-HP for the obvious reason that DA-HP violates a primary policy objective in the Italian teacher assignment, eliminating JE, while DA-HC does not. Nevertheless, we still compare DA-HC and DA-HP on the efficiency grounds to see whether there is any efficiency-fairness trade off. First, theoretically, there is no Pareto dominance relation between DA-HC and DA-HP because, for instance in Example 1, teacher t_1 prefers DA-HC to DA-HP while teacher t_2 prefers DA-HP to DA-HC. On the other hand, in our subsample, we observe that 37.7% of the teachers prefer DA-HP to DA-HC assignment, 0.4% of the teachers prefers DA-HC to DA-HP assignment, while the remaining are indifferent between the two mechanisms. In the DA-HP assignment, 1,310 teachers' priorities (12.9 % of all teachers) are violated in at least one school, while this number is zero in the DA-HC assignment in line with our theoretical results. Our empirical observations

[29]Because of "professional mobility" (e.g., a teacher moving from Maths to Economics), it is in general complicated to consider all school-field pairs for all types of schools. For simplicity we consider only geographic transfers, excluding professional mobility. See Section B.1 for different types of mobility. This simplification renders the different types of schools and fields as independent markets.

[30]Preschools have a lower number of types as opposed to primary-schools, middle-schools, and high-schools since the only distinction is between normal and special educational needs teachers. Hence, it is more convenient (computationally and data preparation) to focus on the preschool teachers.

[31]Our simulation results that we report in Appendix B.4 provide further evidence that DA-HC can potentially bring welfare improvement over the benchmark DA-STB mechanism for more than 10% of the teachers.

suggest an efficiency-fairness trade-off (despite the absence of a Pareto dominance relation) between DA-HP and DA-HC.[32]

Finally, note that in these empirical comparisons we rely on the ROLs that were submitted under a manipulable mechanism. On the other hand, if teachers indeed manipulate, they essentially do so by including singleton schools (or finer regions) above a region (municipality, district, or province) of the school while they would only rank the region itself if they reported truthfully. That is, putting the intricacies of the portfolio-choice problem apart (note that around 73% of the teacher applications do not use all available slots in their ROLs), the frequency of regions in the ROLs typically decrease as teachers manipulate. Therefore, the prevalence of regions in the actually submitted ROLs would likely carry over to the counterfactual scenario where DA-HC or DA-STB is in use.[33]

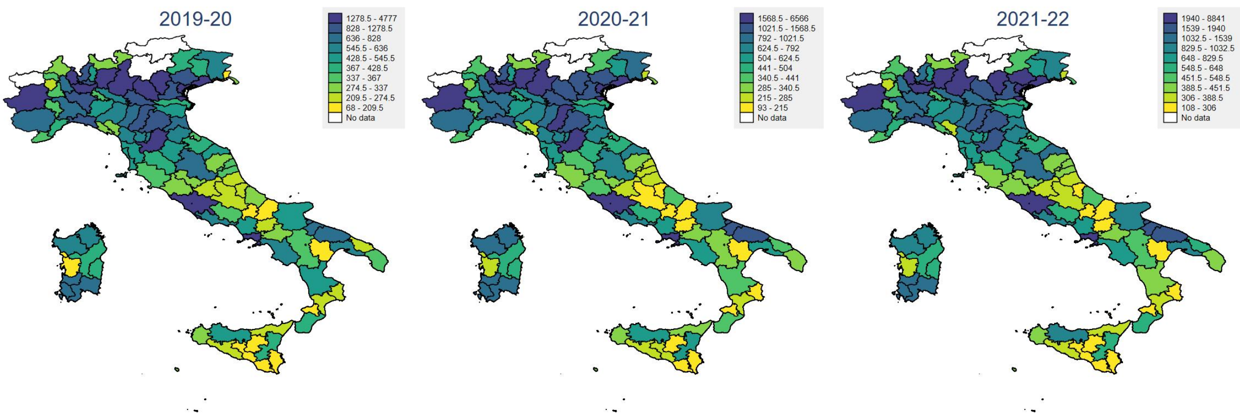
7 Conclusion

Motivated by the Italian teacher assignment, we studied a teacher assignment problem where schools are included in a hierarchical geographical structure. Teachers can rank an entire region, and if they do so, they are considered to be indifferent between all schools within that region. The geographical structure also affects schools' strict priorities. We showed that the current teacher assignment mechanism used in Italy does not eliminate justified envy and is not strategy-proof. We introduced a novel efficiency concept, Pareto-size efficiency subject to eliminating JE, that is suitable for priority-based assignment problems in general when ROL's may include indifferences. We showed that the DA-HC mechanism is optimal in efficiency terms within the class of strategy-proof mechanisms that eliminate justified envy, when indifferences are structured around a hierarchy such as the geographical hierarchy in Italy's teacher assignment system.

[32]Our simulations in Appendix B.4 confirm this efficiency-fairness trade-off, and in particular provide further evidence that DA-HC can eliminate a significant amount of justified envy resulting from DA-HP.

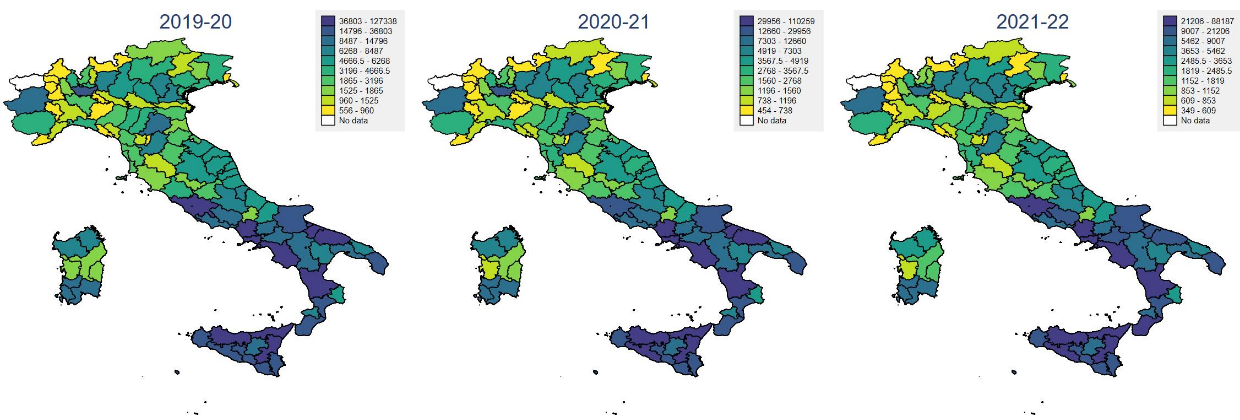
[33]Another direction could be to estimate preferences. However, since the current mechanism is not strategy-proof or stable, a proper modelling of teacher application behavior that accounts for the non-trivial portfolio choice problem becomes crucial. In particular, the current setting provides complex dynamic incentives, due for instance to geographical priorities and to the presence of a temporal constraint (retaining teachers in their current position for a certain period). Additionally, the estimation framework should also be robust to teachers' search, since in the Italian teacher assignment, teachers have incomplete information about available vacancies when they apply. Consequently, the existing estimation strategies, e.g., Agarwal and Somaini (2018, 2020), Calsamiglia et al. (2020), and Fack et al. (2019), cannot be directly used in our setting, and therefore this direction goes beyond the scope of the current paper.

Figure 3: Distribution of remaining vacancies



Notes: These map shows the distribution of the remaining vacancies available for new teachers by each province in each school-year.
Source: Italian Ministry of Education, Restricted Data.

Figure 4: Distribution of applications received by province



Notes: These map shows the distribution of the applications (of any type) received by each province.
Source: Italian Ministry of Education, Restricted Data.

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Appendix A Proofs

A.1 Proof of Proposition 1

Consider the following problem. Let $T = \{t_1, t_2, t_3\}$, $S = \{s_1, s_2, s_3\}$, $G = \{r_1, r_2, \phi = \{r_1, r_2\}\}$ with $r_1 = \{s_1, s_2\}$, $r_2 = \{s_3\}$, $q_{s_1} = q_{s_2} = 1$, $q_{s_3} = 2$, $\omega_t = s_3$ for each $t \in T$, and ROLs and priorities as depicted below.

R_{t_1}	R_{t_2}	R_{t_3}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
r_1	s_1	s_2	t_1	t_1	t_1
			t_3	t_2	t_2
			t_2	t_3	t_3

Suppose that φ is Pareto efficient subject to eliminating JE. In particular, φ is non-wasteful and eliminates JE, which implies that $\varphi_{t_1}(R) \neq \omega_{t_1} = s_3$ since t_1 has top priority at both s_1 and s_2, and finds both schools acceptable.

Consider the case where $\varphi_{t_1}(R) = s_1$. Then, $\varphi_{t_2}(R) = \omega_{t_1} = s_3$ and $\varphi_{t_3}(R) = s_2$. Consider the following misreport by t_2, $R'_{t_2} : s_1, s_2$ (that is, first-ranks s_1 and second-ranks s_2). Now, there is a unique matching that is Pareto efficient subject to eliminating JE, where t_1 gets s_2 and t_2 gets s_1, implying that φ is not strategy-proof since t_2 becomes better off by misreporting her ROL. A symmetrical argument applies for the remaining case where $\varphi_{t_1}(R) = s_2$.

A.2 Proof of Proposition 2

Consider the following problem. Let $T = \{t_1, t_2, t_3\}$, $S = \{s_1, s_2, s_3, s_4\}$, $\mathcal{G} = \{r_1, r_2, \phi = \{r_1, r_2\}\}$ with $r_1 = \{s_1, s_2\}$, $r_2 = \{s_3\}$, $q_{s_1} = q_{s_2} = q_{s_3} = 1$, $q_{s_4} = 3$, $\omega_t = s_4$ for each $t \in T$, and ROLs and priorities as depicted below.

R_{t_1}	R_{t_2}	R_{t_3}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
r_1	s_1	s_2	t_1	t_1	t_1	t_1
			t_3	t_2	t_2	t_2
			t_2	t_3	t_3	t_3

Suppose that φ is non-wasteful and size efficient subject to eliminating JE. In particular, φ is non-wasteful and eliminates JE, which implies that $\varphi(t_1) \neq \omega_{t_1} = s_4$ since t_1 has top priority at both of her acceptable schools, s_1 and s_2.

Consider the case where $\varphi_{t_1}(R) = s_1$. Then, $\varphi_{t_2}(R) = \omega_{t_2} = s_4$ and $\varphi_{t_3}(R) = s_2$. Consider another problem R' where $R'_{t_1} = R_{t_1}$, $R'_{t_2} = R_{t_2}$, $R'_{t_3} : s_2, s_3$. Since φ eliminates JE at R' as well, we must have $\varphi_{t_1}(R') = s_1$ or $\varphi_{t_1}(R') = s_2$. Note that there is a unique maximum size matching that eliminates JE at R', which must be chosen by φ and therefore $\varphi_{t_1}(R') = s_2$, $\varphi_{t_2}(R') = s_1$, and $\varphi_{t_3}(R') = s_3$. But then, φ is not strategy-proof since, when the true ROL profile is R', t_3 becomes better off by misreporting her ROL as R_{t_3}. A symmetrical argument applies for the remaining case where $\varphi(t_1) = s_2$.

A.3 Proof of Theorem 1

A.3.1 Some preliminary results

Let $A \in \mathcal{A}^\phi$ be given and fixed for this section. We will introduce some notation and prove a set of lemmas and propositions. The first lemma is a simple observation that doesn't require a proof, while we will invoke this observation in several proofs.

Lemma 1. *Let (A', A'') be a partition of A and (q', q'') be such that, for each $s \in S$, $q'_s + q''_s = q_s$. Let μ' be a matching for the problem (A', q') and μ'' be a matching for the problem (A'', q''). If there exists a maximum size matching μ for the combined problem (A, q) that coincides with μ' for the teachers with applications in A', and if μ'' is a maximum size matching for (A'', q''), then μ' combined with μ'' is a maximum size matching for the combined problem (A, q).*

We call a pair (A', q') with $A' \subseteq A$ and $q'_s \leq q_s$ for each $s \in S$, a **reduced problem** if there is a way of assigning teachers with applications in $A \setminus A'$ to their feasible schools in which q' is the profile of vacant seats. We next show that, for any reduced problem, there exists a maximum size matching in which the smallest-index teacher among finest applications is accepted to her smallest-index feasible school.[34]

Lemma 2. *Let (A', q') be any reduced problem. There exists a maximum size matching in which the smallest-index teacher among finest applications is accepted to her smallest-index feasible school.*

Proof. Given any reduced problem (A', q'), consider the following choice rule that we call the simple hierarchical choice rule (SHC^ϕ). Note that, in contrast with HC^ϕ, SHC^ϕ is based on a myopic choice procedure that does not take into account the priority orderings.

$$SHC^\phi$$

Step 1*:* Consider the smallest-index teacher among finest applications, say (t, x). Accept (permanently) t to the school in $F(x)$ with the smallest index. Move to the next step.

Steps $k > 1$*:* If there is no vacant seat left or all teachers with an application are considered, then terminate. Otherwise, among the teachers who have not been considered yet, consider the smallest-index teacher among finest applications, say (t, x). Accept (permanently) t to the school in $F(x)$ with the smallest index that has a vacant seat (if any). Move to the next step.

We will show that, for any reduced problem (A', q'), $SHC^\phi(A', q')$ is a maximum size matching: there is no assignment of teachers with applications in A' to their feasible schools that respects schools' reduced capacity constraints q' and assigns more teachers than $SHC^\phi(A', q')$. This will also imply that there exists a maximum size matching in which the smallest-index teacher among finest applications is accepted to her smallest-index feasible school.

Suppose, towards a contradiction, that there is another matching for the reduced problem that has a greater size than $SHC^\phi(A', q')$. Note that for any teacher who is not assigned to any school, none of her feasible schools has any vacant seats in $SHC^\phi(A', q')$ by construction. Then, there exists a list of teachers $t_1, \ldots, t_k$ with $k \geq 2$, and a school s such that t_1 is not assigned to any school in $SHC^\phi(A', q')$, all other teachers in the list are assigned to some schools in $SHC^\phi(A', q')$, and s has a vacant seat in $SHC^\phi(A', q')$, such that the assigned

[34] Note that the proof includes a simple polynomial time algorithm that finds a maximum size matching in our context (where applications are either distinct or nested), while there are already known algorithms in the literature that find a maximum size matching in the general bipartite matching context.

school of t_i is feasible also for t_{i-1} for each $i \in \{1, \dots, k\}$ and s is feasible for t_k, i.e., there exists an *augmenting path* (Berge, 1957). Without loss of generality, assume that $t_1, \dots, t_k$ is a shortest size augmenting path. Note that the assigned school of t_3 is not feasible for t_1 (since otherwise there would be a shorter augmenting path) while it is feasible for t_2. Moreover, the assigned school of t_2 is feasible for both t_1 and t_2. Hence, t_2 has a strictly coarser application than t_1, contradicting that t_2 is considered before t_1 in the SHC^{ϕ} algorithm since t_2 was assigned a school that is feasible for both while t_1 was left with no feasible vacant seat in the step where t_1 was considered. □

We next show that, as we iterate the steps of the $HC^{\phi}(A)$ algorithm, the set of critical teachers is preserved. For any set of plausible applications A and any capacity profile q', we say an applicant t is critical in (A, q') if t is critical in A when the capacity profile is q'.

Lemma 3. *For any Step k of the $HC^{\phi}(A)$ algorithm, let (A^k, q^k) denote the reduced problem at the beginning of Step k, i.e., A^k is the set of applications from teachers who are not tentatively accepted by any school and who have not been rejected by all their feasible schools by the beginning of Step k, and q^k is the profile of vacant seats at the beginning of Step k. Take any teacher t with some $(t, x) \in A^k$. Then, t is critical in (A, q) if and only if t is critical in (A^k, q^k).*

Proof. Suppose, towards a contradiction, that the statement is not true. Without loss of generality, assume that Step k is the earliest step of the algorithm at which the statement is violated. Let t' with $(t', x') \in A$ be the teacher considered in Step $k-1$.

Case 1: Suppose, towards a contradiction, that a teacher t is critical in (A^{k-1}, q^{k-1}) but not critical in (A^k, q^k).

Subcase 1: t' is tentatively accepted to the smallest index school in $F(x')$ that has a vacant seat in Step $k-1$ of the algorithm, say s. Since t is not critical in (A^k, q^k), there exists a maximum size matching for the reduced problem (A^k, q^k), say μ, in which t is not assigned to any school. By Lemma 2, there exists a maximum size matching for the reduced problem (A^{k-1}, q^{k-1}) in which t' is accepted to s. But then, by Lemma 1, the matching obtained from μ by assigning t' to s is a maximum size matching for the reduced problem (A^{k-1}, q^{k-1}), in which t is not assigned to any school, contradicting that t is critical in (A^{k-1}, q^{k-1}).

Subcase 2: t' replaces some teacher t'' with $(t'', x'') \in A$ in some school s in Step $k-1$ of the algorithm. Note that t' is not critical in (A^{k-1}, q^{k-1}).

Suppose that $F(x'') \subseteq F(x')$. Note that t'' is not critical in (A^k, q^k) since $F(x'') \subseteq F(x')$, $q^{k-1} = q^k$, $A^{k-1} = (A^k \setminus \{t''\}) \cup \{t'\}$, and t' is not critical in (A^{k-1}, q^{k-1}). Then, the size of the maximum size matching in (A^{k-1}, q^{k-1}) is the same as the size of the maximum size matching in (A^k, q^k). Since t is not critical in (A^k, q^k), there exists a maximum size matching for the reduced problem (A^k, q^k), say μ, in which t is not assigned to any school. Then, the matching obtained from μ by replacing t' with t'' in s is a maximum size matching for the reduced problem (A^{k-1}, q^{k-1}), in which t is not assigned to any school, contradicting that t is critical in (A^{k-1}, q^{k-1}).

Suppose that $F(x') \subsetneq F(x'')$. Observe that there exists a list of teachers $t_1, \ldots, t_n$ with applications $(t_1, x_1), \ldots, (t_n, x_n) \in A$, a list of schools $s_2, \ldots, s_n$ such that

- $t_{n-1} = t'$ and $t_n = t''$,
- $s_n = s$,
- $F(x_j) \subseteq F(x_1)$ for each $j > 1$, and
- t_1 replaces t_2 in s_2 at some Step $k_1 < k$ of the algorithm, then in the next step t_2 replaces t_3 in s_3, …, then finally $t_{n-1} = t'$ replaces $t_n = t''$ in $s_n = s$ in Step $k-1$ of the algorithm.

Note that t_1 is not critical in (A^{k_1}, q^{k_1}). Also note that t'' is not critical in (A^k, q^k) since $F(x'') \subseteq F(x_1)$, $q^{k_1} = q^k$, $A^{k_1} = (A^k \setminus \{t''\}) \cup \{t_1\}$, and t_1 is not critical in (A^{k_1}, q^{k_1}). Then, the size of the maximum size matching in (A^{k_1}, q^{k_1}) is the same as the size of the maximum size matching in (A^k, q^k). Since t is not critical in (A^k, q^k), there exists a maximum size matching for the reduced problem (A^k, q^k), say μ, in which t is not assigned to any school. Then, the matching obtained from μ by assigning $t_n = t''$ to $s_n = s$, $t_{n-1} = t'$ to s_{n-1},…, t_2 to s_2, and leaving t_1 unassigned, is a maximum size matching for the reduced problem (A^{k_1}, q^{k_1}), in which t is not assigned to any school, contradicting that t is critical in (A^{k_1}, q^{k_1}).

Case 2: Suppose, towards a contradiction, that a teacher t is not critical in (A^{k-1}, q^{k-1}) but critical in (A^k, q^k).

Subcase 1: t' is tentatively accepted to the smallest index school in $F(x')$ that has a vacant seat in Step $k-1$ of the algorithm, say s. Since t is not critical in (A^{k-1}, q^{k-1}), there exists a maximum size matching for the reduced problem (A^{k-1}, q^{k-1}), say μ, in which t is not assigned to any school. Without loss of generality, assume that t' is assigned to s in μ

(otherwise, either we can move t' to a vacant seat in s if there is any vacant seat, or we can switch the schools of t' and another teacher who is assigned s, which is feasible since t' is a teacher with a finest application). Then, the matching obtained from μ by removing t' is a maximum size matching for the reduced problem (A^k, q^k), in which t is not assigned to any school, contradicting that t is critical in (A^k, q^k).

Subcase 2: t' replaces some teacher t'' in some school s in Step $k-1$ of the algorithm. Note that t' is not critical in (A^{k-1}, q^{k-1}).

Suppose that $F(x'') \subseteq F(x')$. Note that t'' is not critical in (A^k, q^k) since $F(x'') \subseteq F(x')$, $q^{k-1} = q^k$, $A^{k-1} = (A^k \setminus \{t''\}) \cup \{t'\}$, and t' is not critical in (A^{k-1}, q^{k-1}). Then, the size of the maximum size matching in (A^{k-1}, q^{k-1}) is the same as the size of the maximum size matching in (A^k, q^k). Since t is not critical in (A^{k-1}, q^{k-1}), there exists a maximum size matching for the reduced problem (A^{k-1}, q^{k-1}), say μ, in which t is not assigned to any school. Then, the matching obtained from μ by replacing t'' with t' in s is a maximum size matching for the reduced problem (A^k, q^k), in which t is not assigned to any school, contradicting that t is critical in (A^k, q^k).

Suppose that $F(x') \subsetneq F(x'')$. As above, there exists a list of teachers $t_1, \ldots, t_n$ with applications $(t_1, x_1), \ldots, (t_n, x_n) \in A$, a list of schools $s_2, \ldots, s_n$ such that

- $t_{n-1} = t'$ and $t_n = t''$,
- $s_n = s$,
- $F(x_j) \subseteq F(x_1)$ for each $j > 1$, and
- t_1 replaces t_2 in s_2 at some Step $k_1 < k$ of the algorithm, then in the next step t_2 replaces t_3 in s_3, ..., then finally $t_{n-1} = t'$ replaces $t_n = t''$ in $s_n = s$ in Step $k-1$ of the algorithm.

Note that t_1 is not critical in (A^{k_1}, q^{k_1}). Also note that t'' is not critical in (A^k, q^k) since $F(x'') \subseteq F(x_1)$, $q^{k_1} = q^k$, $A^{k_1} = (A^k \setminus \{t''\}) \cup \{t_1\}$, and t_1 is not critical in (A^{k_1}, q^{k_1}). Then, the size of the maximum size matching in (A^{k_1}, q^{k_1}) is the same as the size of the maximum size matching in (A^k, q^k). Since t is not critical in (A^{k-1}, q^{k-1}), there exists a maximum size matching for the reduced problem (A^{k-1}, q^{k-1}), say μ, in which t is not assigned to any school. Then, the matching obtained from μ by assigning t_1 to s_2, t_2 to s_3,..., $t_{n-1} = t'$ to $s_n = s$, and leaving t'' unassigned, is a maximum size matching for the reduced problem (A^k, q^k), in which t is not assigned to any school, contradicting that t is critical in (A^k, q^k). □

Lemma 4. *For any Step k of the $HC^{\phi}(A)$ algorithm, let μ^k be the matching at the end of Step k (determined by the tentative acceptances at the end of Step k). There exists a maximum size matching μ for the entire problem (A, q) which coincides with μ^k for the teachers who are tentatively accepted at the end of Step k.*

Proof. For any Step k of the $HC^{\phi}(A)$ algorithm, let (A^k, q^k) denote the reduced problem at the beginning of Step k, i.e., A^k is the set of applications from teachers who are not tentatively accepted by any school and who have not been rejected by all their feasible schools by the beginning of Step k, and q^k is the profile of vacant seats at the beginning of Step k.

We prove by induction. Consider Step 1. Note that μ^1 is such that the smallest-index teacher among finest applications, say t, is assigned to her smallest-indexed feasible school. By Lemma 2, there exists a maximum size matching that coincides with μ^1 for the only teacher who has been considered by the end of Step 1.

Assume that the claim is true for any step before some Step k. Consider Step k. Let teacher t with application (t, x) be the teacher considered in Step k.

Case 1: Suppose that t is critical for maximizing the size of (A, q). By Lemma 3, t is critical for maximizing the size of the reduced problem (A^k, q^k). Then, t is tentatively accepted to the smallest index school in $F(x)$ that has a vacant seat, say s. By the induction assumption, there exists a maximum size matching for the entire problem that coincides with μ^{k-1} for the teachers who have been considered by the end of Step $k-1$. By Lemma 2, there exists a maximum size matching for the reduced problem where t is assigned to s, say μ. Then, by Lemma 1, μ^{k-1} combined with μ, which coincides with μ^k for the teachers who have been considered by the end of Step k, is a maximum size matching for the entire problem (A, q).

Case 2: Suppose that t is not critical for maximizing the size of (A, q). By Lemma 3, t is not critical for maximizing the size of the reduced problem (A^k, q^k). By the induction assumption, there exists a maximum size matching for the entire problem that coincides with μ^{k-1} for the teachers who have been considered by the end of Step $k-1$.

Suppose that t is tentatively accepted to s because s has a vacant seat. By Lemma 2, there exists a maximum size matching for the reduced problem where t is assigned to s, say μ. Then, by Lemma 1, μ^{k-1} combined with μ, which coincides with μ^k for the teachers who have been considered by the end of Step k, is a maximum size matching for the entire problem (A, q).

Suppose that t is tentatively accepted to s by replacing another teacher t'. Since t is not critical for maximizing the size of the reduced problem (A^k, q^k), there exists a maximum size matching μ for the reduced problem where t is not matched to any school. Then, by Lemma 1, μ^{k-1} combined with μ, let us call it μ', is a maximum size matching for the entire problem (A, q). Now, consider the matching μ'' obtained from μ' by just replacing t with t' at school s. Note that this is feasible, μ'' is also a maximum size matching for the entire problem (A, q) and coincides with μ^k for the teachers who have been tentatively accepted by the end of Step k.

Suppose that t is rejected by s. Since t is not critical for maximizing the size of the reduced problem (A^k, q^k), there exists a maximum size matching μ for the reduced problem where t is not matched to any school. Then, by Lemma 1, μ^{k-1} combined with μ, let us call it μ', is a maximum size matching for the entire problem (A, q) and coincides with μ^k for the teachers who have been tentatively accepted by the end of Step k. □

Lemma 5. *Suppose that, at some step of the $HC^{\phi}(A)$ algorithm, a teacher who is critical in (A, q) is assigned to a school s. Then, no teacher is ever rejected, in particular t is never rejected, by s in the course of the $HC^{\phi}(A)$ algorithm.*

Proof. Suppose that t is critical in (A, q). Then, t is assigned a vacant seat at the first step t is considered, say in Step k at school $s \in F(x)$. We claim that no teacher is ever rejected, in particular t is never rejected, by s in the course of the algorithm. Clearly, no teacher is rejected from s until and including Step k. Suppose, towards a contradiction, that there is an earliest step, say Step k', at which a teacher is rejected from s. Then, the teacher t' who is considered at Step k' is not critical in (A, q) or in $(A^{k'}, q^{k'})$, and t' replaces a teacher in s at Step k'. Since k' is not critical in $(A^{k'}, q^{k'})$, by Lemma 1, there exists a maximum size matching for the entire problem (A, q), say μ, in which t' is not assigned to any school and t is assigned to s. But then, the matching obtained from μ by replacing t with t' in s is also a maximum size matching for the entire problem (A, q), contradicting that t is critical in (A, q). □

Lemma 6. *Suppose that $(t, x), (t', x') \in A$ and $t \neq t'$. If t is critical in (A, q), then t is critical also in $(A \setminus \{(t', x')\}, q)$.*

Proof. Suppose, towards a contradiction, that t is not critical in $(A \setminus \{(t', x')\}, q)$. Then, there exists a maximum matching μ for the problem $(A \setminus \{(t', x')\}, q)$ in which t is not assigned to any school. Since t is critical in (A, q), μ is not a maximum size matching in (A, q). Then, there exists a list of teachers $t_1, \ldots, t_k$ and a school s such that t_1 is not assigned to any school in μ, all other teachers in the list are assigned to some schools in μ, and s has a vacant seat in μ, such that $\mu(t_i)$ is feasible also for t_{i-1} for each $i \in \{1, \ldots, k\}$ and s is feasible for t_k, i.e., there exists an *augmenting path* (Berge, 1957). Since μ is a maximum matching for the problem $(A \setminus \{(t', x')\}, q)$, $t_1 = t'$. Let μ' be the matching obtained from μ by assigning t' to s_1 and implementing the reassignment along the augmenting path. Now, observe that the size of μ' is one more than the size of μ, and μ' is a maximum size matching for the problem (A, q) (since (A, q) includes one more teacher compared to $(A \setminus \{(t', x')\}, q)$, their maximum size matchings can differ by at most one in size) where t is not assigned to any school, contradicting that t is critical in (A, q). □

Proposition 3. *HC^ϕ always chooses a maximum size matching. That is, for any $A \subseteq \mathcal{A}^\phi$, there is no assignment of teachers to their feasible schools that respects schools' capacity constraints and assigns more teachers than $HC^\phi(A)$.*

Proof. Directly follows from Lemma 4. □

Proposition 4. *HC^ϕ always eliminates JE. That is, for any $A \subseteq \mathcal{A}^\phi$, if an applicant is not assigned to any school, than all her feasible schools must be filled with teachers who have higher priorities.*

Proof. If $HC_t^\phi(A) = \emptyset$, it must be that t has applied to each school in $F(x)$ in the course of the HC^ϕ algorithm and got rejected from each of them. Note that whenever t is rejected by some $s \in F(x)$ at a step of the HC^ϕ algorithm, all seats in s must be assigned to teachers with higher priorities at that step and thereafter. □

Proposition 5. *HC^ϕ satisfies substitutability in the following sense. For any $A \subseteq \mathcal{A}^\phi$, if a teacher is accepted to a school from a set of applications, the teacher will still be accepted to a school (not necessarily the same school) if any application from another student is removed from the set of applications.*

Proof. Suppose that $(t, x), (t', x') \in A$, $t \neq t'$, and $HC_t^\phi(A) \neq \emptyset$. We want to show that $HC_t^\phi(A \setminus \{(t', x')\}) \neq \emptyset$. Suppose that t is critical in (A, q). Then, by Lemma 6, t is critical also in $(A \setminus \{(t', x')\}, q)$, and by Lemma 5, $HC_t^\phi(A \setminus \{(t', x')\}) \neq \emptyset$.

Suppose that t is not critical in (A, q). Let $HC_t^\phi(A) = s$. Suppose, towards a contradiction, that $HC_t^\phi(A \setminus \{(t', x')\}) = \emptyset$. By Proposition 5, in $HC^\phi(A \setminus \{(t', x')\})$, all seats in s are assigned teachers who have higher priority than t, and by Lemmas 5 and 6, all these teachers are not critical both in (A, q) and $(A \setminus \{(t', x')\}, q)$.

Then, there exists a teacher t^* with $HC_{t^*}^\phi(A \setminus \{(t', x')\}) = s$ such that t^* is assigned to a s' which has a lower index than s, in $HC^\phi(A)$, while t^* is rejected by s' in the $HC^\phi(A \setminus \{(t', x')\})$ algorithm. Then, there exists a teacher t^{**} with $HC_{t^{**}}^\phi(A \setminus \{(t', x')\}) = s'$ such that t^{**} is assigned to a school s'' which has a lower index than s, in $HC^\phi(A)$, while t^{**} is rejected by s' in the $HC^\phi(A \setminus \{(t', x')\})$ algorithm. Continuing similarly, we reach a contradiction since the number of schools is finite. □

A.3.2 Completing the Proof of Theorem 1

In the rest of the proof, we first formulate the problem as a matching with contracts problem (Hatfield and Milgrom, 2005). Then, following Hatfield and Kominers (2019), we define a *pseudo choice function* for each province ϕ, which may sometimes choose two different contracts of the same teacher but coincides with HC^ϕ on the domain of choice problems where each teacher has at most one application (at most one contract). In fact, this pseudo choice function operates the same as HC^ϕ, but treats different contracts of the same teacher as if they included different teachers. Consequently, we obtain a cumulative offers algorithm that coincides with the DA-HC algorithm at each step, and use this coincidence to conclude that the DA-HC mechanism satisfies the desirable properties. Unlike Hatfield and Kominers (2019), the primitive choice rule in our context, HC^ϕ, is not defined on the complete domain of choice problems. However, as recently illustrated in Doğan and Erdil (2022), the results in Hatfield and Kominers (2019) apply in our context as well. Next, we provide the formal proof.

We associate each teacher assignment problem with a matching with contracts problem where each contract (t, ϕ, x) specifies a teacher $t \in T$, a province ϕ, and a school or a region in the province $x \in S^\phi \cup \mathcal{M}^\phi \cup \mathcal{D}^\phi \cup \{\phi\}$. A **matching** is a collection of contracts such that each student appears in at most one contract. A **pseudo matching** is simply a collection of contracts, i.e., a student might appear in several contracts unlike a matching.

Let $\mathcal{X}$ be the set of all possible contracts and, for each province ϕ, let $\mathcal{X}_\phi \subseteq \mathcal{X}$ denote the set of contracts that include province ϕ. Each $X \subseteq \mathcal{X}_\phi$ is called a **choice problem for**

province ϕ. A **choice function** of the province ϕ associates each choice problem $X \subseteq \mathcal{X}_\phi$ with a matching. A **pseudo choice function** of the province ϕ associates each choice problem $X \subseteq \mathcal{X}_\phi$ with a pseudo matching.

For each province ϕ, index the contracts in $\mathcal{X}_\phi$ in a way that is consistent with the indexes of the teachers, i.e., for any pair of contracts including teachers t and t' such that $t \neq t'$, the contract including t has a lower index than the contract including t' if and only if t has a lower index than t'. Also, create auxiliary teachers $T^{aux} = t^1, \ldots, t^{|\mathcal{X}_\phi|}$ and associate the lowest index auxiliary teacher with the lowest index contract, the second-lowest index auxiliary teacher with the second-lowest index contract, and so on. For each school $s \in S$, define an auxiliary priority ordering $\succ_s^{aux}$ over T^{aux} that is consistent with $\succ_s$. That is, for any $t_1^{aux}, t_2^{aux} \in T^{aux}$ and $t, t' \in T$ such that t and t' are the teachers in the contracts associated with t_1^{aux} and t_2^{aux}, respectively, if $t \succ_s t'$, then $t_1^{aux} \succ_s^{aux} t_2^{aux}$.

Now, for each province ϕ, define its pseudo choice function Ch_ϕ as follows. Take any $X \subseteq \mathcal{X}_\phi$. Define an auxiliary set of applications A^{aux}, induced by X, as follows: there exists a contract $(t, \phi, x) \in X$ if and only if there exists $(t^{aux}, x) \in A^{aux}$ where t^{aux} is the auxiliary teacher associated with the contract (t, ϕ, x). Finally, for each $(t, \phi, x) \in X$, $(t, \phi, x) \in Ch_\phi(X)$ if and only if the auxiliary teacher t^{aux} associated with (t, ϕ, x) is assigned a school in $HC^\phi(A^{aux})$ (in the auxiliary problem with T^{aux} and $\succ^{aux}$).

It is easy to see that Ch_ϕ satisfies the following properties.

Substitutes: If a contract is chosen given a choice problem, then it is still chosen if another contract is removed from the choice problem. This directly follows from Proposition 5.

Law of aggregate demand (LAD): If a contract is removed from a choice problem, then the number of chosen contracts does not increase. This directly follows from Proposition 3.

Irrelevance of rejected contracts (IRC): If a rejected contract is removed from a choice problem, then the set of chosen contracts remains the same. This is because substitutes together and LAD imply IRC (Aygün and Sönmez, 2013).

We next define the cumulative offers (CO) algorithm.

Cumulative Offers (CO) Algorithm

Step 1: Each teacher proposes their top-ranked acceptable contract. If there is no proposal, then stop and return the resulting (empty) matching. Otherwise, let each province ϕ hold the contracts that its pseudo choice function Ch_ϕ chooses from those that have been proposed to the province, and go to Step 2.

Step $k \geq 2$: Each teacher who is not included in a currently held contract proposes their next-best acceptable contract. If there is no proposal, then stop and return the pseudo matching which consists of the contracts held by the provinces at the end of the previous step. Otherwise, let each province ϕ hold the applications that its pseudo choice function Ch_{ϕ} chooses from the cumulative set of all proposals that the province has received since the beginning of Step 1, and go to Step $k+1$.

The algorithm must eventually stop because no teacher proposes the same contract twice.

The Cumulative Offers (CO) mechanism returns, for each problem, the pseudo matching produced by the CO algorithm. The lemma below will establishes the equivalence of DA-HC and CO, and in particular verify that the outcome of CO is actually a matching. The **tentative assignment** at the end of Step k is the pseudo matching where each province is matched with the applications it is holding at the end of that step.

Lemma 7. *The tentative assignments of the DA-HC mechanism and the CO mechanism coincide at every step. That is, for every $k \geq 1$, a teacher t applies to item (a school or a region) x and is tentatively assigned to a school $s \in x$ in province ϕ at the end of Step k in the DA-HC algorithm if and only if the contract (t, ϕ, x) with $s \in x$ is tentatively accepted in Step k of the CO algorithm.*

Proof. Given an arbitrary problem, consider the first steps of the DA-HC and CO algorithms. Note that, teacher t applies to item x of province ϕ in Step 1 of the DA-HC algorithm if and only if t proposes (i, ϕ, x) in Step 1 of the CO algorithm. Since no teacher applies to more than one region at this step, and since HC^{ϕ} coincide with the associated pseudo choice function whenever no teacher applies to more than one region, the tentative assignments of the DA-HC and CO mechanisms coincide at Step 1.

Suppose the statement holds up to some Step $k-1$. Consider Step k of the DA-HC and CO algorithms. Note that, teacher t is applies to x in province ϕin Step k of the DA-HC algorithm if and only if i proposes (i, ϕ, x) in Step k of the CO algorithm. Now, remember that in the CO algorithm, each province holds the application that its pseudo choice function chooses from those that have been proposed to the province since the beginning of Step 1. If a teacher has proposed more than one application to the province ϕ, it must be that all those applications except for the one that sheproposed last, must have been rejected by ϕ's pseudo choice function in the previous steps. Throughout the CO algorithm, the set of proposed applications is weakly expanding for each province. Therefore, IRA implies that the set of applications chosen by ϕ's pseudo choice function from those that have been proposed to ϕ since the beginning of Step 1 is the same as the set of applications chosen from those that have been proposed since the beginning of Step 1 and not yet rejected. Note that, for each province, applicants from Step k together with tentatively accepted applicants to its courses in the DA-HC algorithm coincides with the set of applications that have been proposed and never rejected since the beginning of Step 1 in the CO algorithm. Also note that in Step k, no teacher is included in two different applications among those that have been proposed to ϕ and not yet rejected since the beginning of Step 1 of the CO algorithm. Since HC^{ϕ} coincide with the associated pseudo choice function whenever no teacher applies to more than one region, the tentative assignments of the DA-HC and CO mechanisms coincide at Step k. □

The fact that DA-HC is strategy-proof directly follows from Hatfield and Kominers (2019), Theorem 3.

DA-HC is non-wasteful and Pareto-size efficient subject to eliminating JE

Consider an arbitrary problem and let μ be the outcome of the DA-HC mechanism for this problem. Take any teacher t and school s such that t prefers s to her assigned school (s might be included in a region ranked by t above her assignment). Let ϕ be the province including s. Let X be the set of contracts considered by ϕ in the last step of the CO algorithm. Note that $(t, \phi, x) \in X$ for some region x with $s \in x$ (or $s = x$) since t must have applied to s in the course of the DA-HC algorithm and DA-HC and CO mechanisms coincide at every step. Since HC^{ϕ} always chooses a maximum size matching by Proposition 3, Ch_{ϕ} always chooses a maximum size matching as well, and therefore all seats in s must be assigned, implying that DA-HC is non-wasteful. Moreover, since HC^{ϕ} eliminates JE by Proposition 4, $Ch_{\phi}(X)$ also eliminates JE, and therefore all seats in s must be filled with teachers who have higher priority than t. Hence, DA-HC eliminates JE.

Suppose, towards a contradiction, that μ is not Pareto-size efficient subject to eliminating JE. Let us call a list of m teachers $(t_0, t_1, \ldots, t_{m-1})$ and m schools $(s_1, \ldots, s_m)$ an **intra-province improvement path** at μ if all schools in $\{s_1, \ldots, s_m\}$ belong to the same province and

i. $s_1 \; \overline{P}_{t_0} \; \mu(t_0)$,

ii. for each $i \in \{1, \ldots, m-1\}$, $s_{i+1} \; \overline{I}_{t_i} \; s_i = \mu(i)$, and

iii. $|\mu(s_m)| < q_{s_m}$.

The following lemma will be useful.

Lemma 8. *Given a problem, if a non-wasteful matching μ eliminates justified envy but it is not Pareto size-efficient subject to eliminating justified envy, then there exists an intra-province improvement path at μ.*

Proof. Let μ' be another matching that also eliminates justified envy, Pareto dominates μ, and matches more teachers with acceptable schools. Then, there exists a teacher, say t_0, who is not assigned a school at μ but assigned a school in μ', say school s_1.

Since μ is non-wasteful and eliminates justified envy, all seats in s_1 must be assigned to teachers with higher priority than t_0 at μ. Since μ' also eliminates justified envy, there exists a teacher, say t_1, who is assigned to s_1 in μ, and assigned to a weakly better but different school at μ', say s_2. Without loss of generality, we can assume $s_2 \; I_{t_1} \; s_1$, since we could as well use t_1 instead of t_0. Also, since a teacher is indifferent between s_1 and s_2, there exists a region including both s_1 and s_2.

If $|\mu(s_2)| < q_{s_2}$, then we are done. Otherwise, there exists a teacher, say t_2, who is assigned to s_2 in μ, and assigned a weakly better but different school at μ', say s_3. Without loss of generality, we can assume $s_3 \; I_{t_2} \; s_2$, since we could as well use t_1 instead of t_0. Also, since a teacher is indifferent between s_2 and s_3, there exists a region including both s_2 and s_3, and since there exists a region including s_1 and s_2, there exists also a region including s_1, s_2, and s_3.

If $|\mu(s_3)| < q_{s_3}$, then we are done. Otherwise, we continue similarly, and since the number of students is finite, we must eventually reach the desired conclusion. □

Now, by Lemma 8, there exists a list of m teachers $(t_0, t_1, \ldots, t_{m-1})$ and m schools $(s_1, \ldots, s_m)$ such that all schools in $\{s_1, \ldots, s_m\}$ belong to the same province, say ϕ, and

i. $s_1 \; P_{t_0} \; \mu(t_0)$,

ii. for each $i \in \{1, \ldots, m-1\}$, $s_{i+1}\ I_{t_i}\ s_i = \mu(i)$, and

iii. $|\mu(s_m)| < q_{s_m}$.

Let X be the set of contracts considered by ϕ in the last step of the CO algorithm. Note that $(t_0, \phi, x_0), (t_1, \phi, x_1), \ldots, (t_{m-1}, \phi, x_{m-1}) \in X$ for some $x_0, \ldots, x_{m-1}$ such that $s_1 \in x_0$, $s_1, s_2 \in x_1$, …, $s_{m-1}, s_m \in x_{m-1}$. But then, $Ch_\phi(X)$ is not a maximum size matching in X, which is a contradiction since HC^ϕ always chooses a maximum size matching by Proposition 3 and therefore $Ch_\phi(X)$ must also be a maximum size matching in X.

A.4 Proof of Theorem 2

Consider an arbitrary problem R. We claim that no teacher is ever rejected by her DA-STB assignment at R in the course of the DA-HC algorithm at R. Towards a contradiction, suppose this is not true. Let k be the earliest step of the DA-HC algorithm at R at which a teacher, say $t \in T$, is rejected by her DA-STB assignment, say $s \in S$ in province ϕ. Then, there exists a step, say Step p, of the HC^ϕ algorithm that runs in Step k of the DA-HC algorithm at R where t is rejected by s. Without loss of generality, assume that before Step p of the HC^ϕ algorithm that runs in Step k of the DA-HC algorithm at R, no teacher is rejected by her DA-STB assignment at R.

Since t is rejected by s, all seats of s must be assigned to higher $\succ_s$-priority teachers at Step p of the HC^ϕ algorithm that runs in Step k of the DA-HC algorithm at R, and moreover, at least one of those teachers, say $t' \in T$ with $t' \succ_s t$, must be assigned to a different school than s, say $s' \in S$, in the DA-STB assignment at R. By our "earliest step" assumptions, t' is not rejected by s' either before Step k of the DA-HC algorithm at R or before Step p of the HC^ϕ algorithm that runs in Step k of the DA-HC algorithm at R. Therefore, t' must be indifferent between s and s', and in particular in the HC^ϕ algorithm that runs in Step k of the DA-HC algorithm at R, t' must have an application to a region in ϕ that includes both s and s'.

Case 1: s' has a lower index than s. Since t' is not rejected by s' before Step p of the HC^ϕ algorithm that runs in Step k of the DA-HC algorithm at R, and in particular t' applies to s before applying to s', it must be that t' is identified as a critical teacher and s is identified as the smallest index school in $F(x')$ that has a vacant seat, in the HC^ϕ algorithm that runs in

Step k of the DA-HC algorithm at R. But this contradicts to Lemma 5 since t is rejected by s' at a later step of the HC^{ϕ} algorithm that runs in Step k of the DA-HC algorithm at R.

Case 2: s' has a higher index than s. Then, in the DA-STB algorithm at R, t' applies to s before applying to s', but gets rejected at some step of the DA-STB algorithm at R. But then, from that step on, all seats of s must be assigned to higher $\succ_s$-priority teachers in the DA-STB algorithm at R, contradicting that t is assigned to s in the DA-STB assignment at R and $t' \succ_s t$.

Appendix B Supplementary Appendix

In this section, we provide (1) a detailed overview of the teacher assignment system in Italy with references to the specific official sources, (2) a detailed explanation of data and descriptives that underlie our empirical observations in Section 6, and (3) simulation results that provide further evidence for our empirical observations in Section 6.

B.1 An Overview of Teacher Assignment in Italy

In the Italian public school system, there are around 900,000 teachers (Table 4). Teachers belong to two different categories in terms of their types of contract: tenured teachers with permanent positions and untenured teachers with fixed-term contracts. Approximately 80% of teachers have a tenured position (Table 5).

The public teacher labor market in Italy is mainly organized as a centralized market. The recruitment process of tenured teachers involves the achievement of a qualifying certification and a national-level examination.[35] Wages and salary scale are fixed and determined at the state level, such that there is no bargaining at the individual level.

The assignment to school positions is conducted in a nationwide centralized matching procedure. The assignment to entry-level positions and the reassignment of teaching positions to tenured teachers are both conducted through (distinct) centralized matching procedures. While the two assignment mechanisms share some similarities (for example the possibility to list both single schools and regions), in this paper we focus on the reassignment of tenured teachers.

The current reassignment system for tenured teachers is regulated by a national collective contract which is the result of a national collective bargaining agreement between the government and teachers' unions (*Contratto Collettivo Nazionale di Lavoro*, containing the general principles, integrated by specific rules in the *Contratto Collettivo Nazionale Integrativo*). Details on the implementing rules are then enacted in a Ministerial Decree (*Ordinanza sulla mobilità personale docente, educativo ed ATA*). These rules are periodically revised.

Each year, 10-15% of the tenured teachers apply to be reallocated to a more desirable position, however 40% of them fail to do so and remain in their current position (Table 7).

[35] Similarly, the recruitment of untenured teachers is also centralized, at the province level (for instance, *Graduatorie provinciali di supplenza*), though there have been some attempts to decentralize this process (such as the possibility of a direct contact with school principals, *"chiamata diretta" dei Dirigenti scolastici*).

The application period is generally around March-April, where teachers have approximately 15-25 days to submit their application(s), and the outcome of the mechanism is announced around the end of May-June. Teachers can submit their applications online, through the website of the Ministry of Education (under the section *Istanze on line*), and they are also allowed to modify their choices before the deadline.

Teachers can move "geographically", namely from one school to another, or "professionally", between types of school or fields (Table 1). With the geographical mobility teachers can move (i) within the same municipality, (ii) within the same province but between different municipalities, or (iii) between different provinces. With the professional mobility teachers can move (i) between different types of school (for instance, from a preschool to a primary school), or (ii) between different fields (for instance, a *Biology* teacher can move in a position for teaching *Maths*). The same teacher can apply to both geographic mobility and professional mobility. Moreover, teachers can ask both normal and special educational needs (SEN) teaching positions, indicating a preference order between the two types of positions.

Approximately 80-85% of the teachers apply for geographic mobility. There are no application restrictions to the type of geographic mobility. The same teacher, in the same application, can list destinations within the same municipality of their initial assignment, between different municipalities within the same province of their initial assignment, or between different provinces.

While it eventually amounts to the DA-HP mechanism, the assignment process is explained in the official documents through three phases:[36]

- Phase 1: Geographical transfers within the same municipality of the ownership school
- Phase 2: Geographical transfers within the same province of the ownership school (but between different municipalities)
- Phase 3: Geographical transfers between different provinces, and professional mobility.[37]

The mechanism allocates seats following phases from 1 to 3, which gives rise to what we call "geographical priorities", which are one of the five elements defining school priorities.

[36] Art.6, paragraph 2 of CCNI

[37] At the end of the second phase, 50% of the remaining vacancies are allocated to new teachers, who are assigned separately within a different matching procedure. Thus, the available vacancies for Round 3 are only half of the vacancies at the end of Round 2. We do not take this detail into account in our model and in the definitions of the mechanisms for the sake of simplicity.

We used this term to reflect the intrinsic hierarchical geographical structure of the movements within the phases, where teachers moving within the same municipality of their current assignment are given priority over teachers coming from outside the municipality, and teachers moving within the same province of their current assignment are given priority over teachers coming from outside the province. However, within each phase, assignments are determined according to more precise rules, which makes the construction of schools' priority orderings rather complex. In the following section, we provide a description of these rules.[38]

B.1.1 Preliminary operations

Some operations are conducted before the main phases of the mechanism, because they are related to particular circumstances that deserve special consideration. Consequently, the assignment of these positions has a priority over any other one. We report a detailed list and a brief explanation for each of them:

1. For organizational reasons, some schools can be subject to aggregation and merging with other schools, or suppression, in order to have an "optimal" population of students. Therefore, each year some teachers can be moved to a new school complex because of these reorganizations. In the next year, these teachers are allowed to be reallocated with priority into the new school building where they have been moved in due course.

2. Some teachers can be temporarily be assigned to a different role (*fuori ruolo*). These teachers have the right to be reallocated with priority into their original position within the next 5 years.

3. Teachers can be temporarily assigned to a position (*Docenti in Utilizzazione*). If they have been moved for at least 2 years to a teaching position in prison schools, they have the right to be allocated with priority to this position.

4. The Ministry of Education, at the request of the Department of Public Security, may allow the transfer or the temporary assignment (even in another province) of teachers subject to special security measures, like protection in case of gender-based violence or for exceptional reasons of personal safety.

[38] A more exhaustive description can be found in the Appendix of the CCNI.

5. Only for the school year 2019-20, the transfers of teachers to the new fields in "Music High Schools" (*Licei Musicali*) are prioritized.

6. Some tenured teachers currently employed in a specific school-type but, who in the past had a position in another school level of education, can apply to return to their original role.

7. Teachers who had been forced to move into another position because of the suppression of their position can be reallocated to their original position, if the position becomes available again (*Rettifica di titolarità*).

B.1.2 Geographical transfers within the same municipality (Phase 1)

This phase is made by as many movements as municipalities, and concerns all transfers within the same municipality of the current position of the teacher. Movements follow this order:

1. Only in the primary school: transfers between positions (English or regular teaching positions) in the same school complex (*circolo o istituto comprensivo di titolarità*).

2. Transfers of teachers who benefit from the priorities given by the law because of disability and particular health conditions (*First point, art. 13 CCNI*). For these teachers it does not matter whether they come from the initial municipality or not. This type of transfers include any between- or within-province geographical transfer.

3. Transfers of those who have previously been moved, in the last eight years, not voluntarily but because forced by the law, and who ask to move again to their previous position, in the original school or school complex (*Second point, art. 13 CCNI*).

4. Only for high schools, transfers in the same school between daily and evening teaching positions.

5. Transfers of teachers with special priorities due to disability and need of continuous care (*Third point, art. 13 CCNI*).

6. Transfers of teachers with special priorities due to assistance to a child with disability (*Fourth point, art. 13 CCNI*). This applies to the case of municipalities with more districts (i.e. only to municipalities that are metropolitan cities).

7. Transfers of teachers with special priorities due to assistance to a spouse or parent with a disability (*Fourth point, art. 13 CCNI*). This applies to the case of municipalities with more districts (i.e. only to municipalities that are metropolitan cities).

8. Transfers of teachers with priorities given by: 1) at least three years of teaching in hospital or prison schools; 2) at least three years of teaching in adult education or evening courses.

9. Other transfers.

10. Transfers of teachers who must move obligatorily, but have not submitted an application, or they have submitted it, but were not reassigned.

11. Transfers of teachers who have been moved by law, in the last eight years, and asked to move again to their previous municipality (*Fifth point, art. 13 CCNI*).

B.1.3 Geographical transfers within the same province but between different municipalities (Phase 2)

This phase concerns all transfers within the same province of the current position but between different municipalities. Movements follow this order:

1. Transfers of teachers moved by law, but have not submitted any application, or they have submitted it, but were not reassigned yet. The assignment is made considering, among the available positions, the nearest to the previous position of the teacher (i.e. the school where they have their ownership)

2. Transfers of teachers with a disability needing long-term care (*Third point, art. 13 CCNI*).

3. Transfers of teachers asking to move to provide care to a child or someone for whom they have a legal custody (*Fourth point, art. 13 CCNI*).

4. Transfers of teachers asking to move to provide care to the spouse, or a parent with disability (*Fourth point, art. 13 CCNI*).

5. Transfers of teachers with at least three years of teaching in hospital or prison schools.

6. Transfers of teachers with at least three years of teaching in adult education or evening courses.

7. Transfers of teachers whose spouse serves in the Army (*Sixth point, art. 13 CCNI*).

8. Transfers of teachers holding a public office in the local administration (*Seventh point, art. 13 CCNI*).

9. All transfers by teachers who have their school ownership in the province.

10. All teachers (without any priority) who ask to transfer from a special education teaching position to a normal education teaching position (even for transfers in the same municipality).

11. Transfers of teachers who must move obligatorily, and have not been reassigned in previous rounds.

12. Voluntary transfers from teachers whose initial position is in a province subject to administrative changes (*Art. 18 bis CCNI*).

B.1.4 Geographical transfers between different provinces and professional mobility (Phase 3)

This phase concerns all geographical transfers between different provinces and professional mobility (transfers between different fields or type of schools). Movements follow this order:

1. Within or between province transfers across different fields for teachers who benefit from priorities because of disability and particular health conditions (First point, art. 13 CCNI).

2. Within or between province transfers across different types of schools for teachers who benefit from priorities because of disability and particular health conditions (*First point, art. 13 CCNI*).

3. Transfers across different fields for teachers whose field has been suppressed.

4. Transfers across different types of schools for teachers whose field has been suppressed.

5. Transfers across different fields for teachers who in the previous year have taught in a different field than their entitlement.
6. Transfers across different types of schools for teachers who in the previous year have taught in a different field than their entitlement.
7. Transfers across different fields for teachers who do not benefit from any priority.
8. Transfers across different types of schools for teachers who do not benefit from any priority.
9. Between provinces transfers for teachers with a disability who need long-term care (*Third point, art. 13 CCNI*).
10. Between provinces transfers for teachers who ask to move to provide care to a child or someone for whom they have a legal custody with a disability (*Fourth point, art. 13 CCNI*).
11. Between provinces transfers for teachers who ask to move to provide care to the spouse (*Fourth point, art. 13 CCNI*).
12. Between provinces transfers for teachers whose spouse serves in the Army (*Sixth point, art. 13 CCNI*).
13. Transfers for teachers who hold a public office in the local administration (*Seventh point, art. 13 CCNI*).
14. Transfers for teachers who resume their duty, after a trade union leave (*Eighth point, art. 13 CCNI*).
15. Transfers of teachers with at least three years of teaching in hospital or prison schools; transfers of teachers with at least three years of teaching in adult education or evening courses.
16. Between provinces transfers for teachers who do not benefit from any priority.
17. Mandatory transfers for new teachers in 2018-19 that have not obtained an ownership position.

B.1.5 Construction of school priorities

We define schools as the entire school-complex. This definition allows us to refer only to the schools that can be listed by teachers in their application (*Plesso sede di organico*). In fact, teachers cannot list each specific institution within the entire school-complex, but only the principal school-complex that is specifically designated as listable.[39]

We consider four levels of education: preschools, primary schools, middle schools and high schools. High schools can be classified into three broad categories: academic high schools (*Licei*), technical high schools (*Istituti Tecnici*), and vocational high schools (*Istituti Professionali*).

For each level of education we distinguish the respective teaching fields. In fact, teachers can request a position only if they have a specific qualification to teach in that field. Thus, the unit of analysis that we consider is the pair school-type. For preschools we consider 10 types, for middle school 43 types, for high school 156 types.

B.2 Teachers' scores

In this section we report a detailed list of the reasons providing teachers' scores in geographical and professional mobility.

Table 2: Geographical Mobility

Description	*Score*
Seniority	
A) For each year of teaching	6
A1) For each year of teaching in schools in small islands (in addition to the scores in A)	6
B) For each year of teaching before the nomination in the permanent position	
for voluntary mobility	6
for mandatory mobility	3
B1) For each year of teaching before the nomination in the permanent position	
in a pre-school in small islands (in addition to the scores in B)	
for voluntary mobility	6
for mandatory mobility	3
B2) Only for primary school teachers: for each year of teaching	

[39] Art. 9, CCNI. To identify them, we use specific school denominations as in the Official List published by the Ministry of Education (*Bollettini Ufficiali*).

Table 2 Continued: Geographical Mobility

in the position of a special teacher in foreign languages	
(for teachers who *only* teach the foreign language)	
from the school year 1992-93 to the school year 1997-98	
if the teaching happened in the school of nomination	0.5
if the teaching happened outside of the school of nomination	1
C) For three consecutive years of teaching, in the role of permanent teacher,	
in the current school (or analogous definitions)	6
For each further year, within five years	2
For each further year, beyond five years	3
For cases in letter C), scores are counted twice if teaching in small islands	
C1) Only for primary school teachers: for three consecutive years of teaching	
in the position of a special teacher in foreign languages	
(for teachers who *also* teach the foreign language)	
from the school year 1992-93 to the school year 1997-98	
(in addition to the scores in A, A1, B, B2, C)	1.5
Only for primary school teachers: for three consecutive years of teaching	
in the position of a special teacher in foreign languages	
(for teachers who *only* teach the foreign language)	
from the school year 1992-93 to the school year 1997-98	
(in addition to the scores in A, A1, B, B2, C)	3
D) Teachers who, for three years, from the school year 2000-01 to the school year 2007-08,	
have not requested a movement between provinces have an additional one-off score of	10
Family Reasons	
A) For reunification to the spouse, or to the parents or to the children	6
B) For each child of age less than 6	4
C) For each child of age greater than 6 and less than 18,	3
or if greater than 18 when they are unable to work	
D) For the assistance of a child with a physical or psychiatric disability,	6
or addicted to drugs, or the assistance of the spouse or a parent unable to work	
Education and Qualifications	
A) For passing a specific public competitive examination for teaching	12
based on qualifications and exams	
B) For each postgraduate qualification or specialization	5
C) For each university degree beyond the required title to teach in the requested role	3
D) For each advanced course with a duration of at least 1 year	1
E) For each four-year degree or master degree or second-level academic degree	5

Table 2 Continued: Geographical Mobility

beyond the required title to teach in the requested role	
F) For holding a PhD degree	5
G) For primary education only, for each training-advanced course	1
in linguistics and language teaching	
H) For each participation, before the school year 2000-01, in the examination board	1
of a high school graduation examination	
I) For each advanced course for content and language integrated learning to teach	1
a non-linguistic subject in a foreign language (with a requirement at the C1 CEFR level)	
L) For each advanced course for content and language integrated learning	0.5
with no requirement at the C1 CEFR level	
Scores from qualifications in B), C), D), E), F), G), I), and L)	
can be cumulated until a maximum of	10

Table 3: Professional Mobility

Description	*Score*
Seniority	
A) For each year of teaching	6
A1) For each year of teaching in schools in small islands (in addition to the scores in A)	6
B) For each year of teaching before the nomination in the permanent position	
and for each year of teaching before the nomination in the permanent position	
in the pre-school	6
B1) For each year of teaching before the nomination in the permanent position	
in a pre-school in small islands (in addition to the scores in B)	6
B2) Only for primary school teachers: for each year of teaching	
in the position of a special teacher in foreign languages	
(for teachers who *only* teach the foreign language)	
from the school year 1992-93 to the school year 1997-98	
if the teaching happened in the school of nomination	0.5
if the teaching happened outside of the school of nomination	1
C) For three consecutive years of teaching, in the role of permanent teacher,	
in the current school (or analogous definitions)	6
For each further year, within five years	2
For each further year, beyond five years	3
For cases in letter C), scores are counted twice if teaching in small islands	

Table 3 Continued: Professional Mobility

C1) Only for primary school teachers: for three consecutive years of teaching	
in the position of a special teacher in foreign languages	
(for teachers who *also* teach the foreign language)	
from the school year 1992-93 to the school year 1997-98	
(in addition to the scores in A, A1, B, B2, C)	1.5
Only for primary school teachers: for three consecutive years of teaching	
in the position of a special teacher in foreign languages	
(for teachers who *only* teach the foreign language)	
from the school year 1992-93 to the school year 1997-98	
(in addition to the scores in A, A1, B, B2, C)	3
D) Teachers who, for three years, from the school year 2000-01 to the school year 2007-08,	
have not requested a movement between provinces have an additional one-off score of	10
Education and Qualifications	
A) For passing a specific public competitive examination	
based on qualifications and exams for teaching	
in the current role or a higher role	12
B) For passing other public competitive examinations	
based on qualifications and exams for teaching	
in roles of the same level or higher than the current role	6
C) For each postgraduate qualification or specialization	5
D) For each university degree beyond the required title to teach in the requested role	3
E) For each advanced course with a duration of at least 1 year	1
F) For each four-year degree or master degree or second-level academic degree	
beyond the required title to teach in the requested role	6
G) For holding a PhD degree	6
H) For primary education only, for each training-advanced course	1
in linguistics and language teaching	
I) For each participation, before the school year 2000-01, in the examination board	1
of a high school graduation examination	
L) For each year of teaching (or a period beyond 180 days) in the same role requested	3
M) For each advanced course for content and language integrated learning to teach	1
a non-linguistic subject in a foreign language (with a requirement at the C1 CEFR level)	
N) For each advanced course for content and language integrated learning	0.5
with no requirement at the C1 CEFR level	

B.2.1 Special Priorities

There is a system of special priorities, such that teachers who belong to specific categories might gain additional priority. These special priorities are:

1. Disability and particular health conditions. Among these, the highest priority is given in particular to those who have a blindness condition, or need a regular haemodialysis treatment.

2. Teachers who have been moved, because forced by the law, in the last eight years, who ask to move again to their previous position.

3. Teachers with a disability who need long-term care.

4. Teachers who ask to move to provide care to the spouse, or to a child, with a disability; Teachers (identified as the only reference) who ask to move to provide care to a parent with disability; Teachers who move to provide care to someone for whom they have a legal custody.

5. Teachers who have been moved, because forced by the law, in the last eight years, who ask to move to the municipality where there was their previous position.

6. Teachers whose spouse serves in the Army.

7. Teachers who hold a public office in the local administration.

8. Teachers who resume their duty, after a trade union leave (regulated by *C.C.N.Q., 4 December 2017*).

B.3 Data and Descriptives

We use administrative data provided by the Italian Ministry of Education on the universe of applications for the teacher reassignment procedure in the school-years 2019-20, 2020-21, and 2021-22. Data include different datasets, which we merge for the analysis.

i. *Teacher applications*. A dataset containing information about the type of the application, the teacher's score and, where applicable, teacher's special priorities.

Table 4: Total Teachers by Type of School

School Type	*2018-19*	*2019-20*	*2020-21*	*2021-22*	*2022-23*
Preschool	11.58%	11.52%	11.20%	11.13%	11.11%
Primary School	32.03%	32.08%	32.20%	32.31%	32.65%
Middle School	22.41%	22.28%	22.29%	22.19%	22.08%
High School	33.98%	34.12%	34.31%	34.38%	34.16%
Total Teachers	886,175	902,487	907,929	923,854	943,681

Notes: This table reports the total population of teachers in Italy and the corresponding fraction by school type (preschools, primary schools, middle schools, and high schools). The numbers include both tenured and untenured teachers, and both normal education and SEN teachers. We may note a slight increase of the total teachers across years. We report the last 5 years of available data.
Source: Italian Ministry of Education, Open Data.

Table 5: Teachers by contractual type

Contractual Type	*2018-19*	*2019-20*	*2020-21*	*2021-22*	*2022-23*
Untenured	18.47%	20.61%	23.39%	24.35%	25.60%
Tenured	81.53%	79.39%	76.61%	75.65%	74.40%
Total Teachers	886,175	902,487	907,929	923,854	943,681

Notes: This table reports the fraction of tenured and untenured teachers on the total population of teachers in Italy. We may note a decrease of the fraction of tenured contractual teachers over years. We report the last 5 years of available data.
Source: Italian Ministry of Education, Open Data.

Table 6: Teachers and applications

	2019-20	*2020-21*	*2021-22*
Teachers Moving	66,296	56,732	48,802
Total Applications	129,803	108,677	87,454
Total Applicants	115,534	96,577	78,232

Notes: This table reports the total number of tenured teachers applying for mobility, the number of applications, and the number of teachers whose application is successful. We report data for school years from 2019-20 to 2021-22.
Source: Italian Ministry of Education, Restricted Data.

Table 7: Applications by Type

Application Type	*2019-20*	*2020-21*	*2021-22*
Geographical Mobility	83.04%	83.10%	82.14%
Professional Mobility			
Mobility between fields	3.77%	3.67%	3.57%
Mobility between school types	13.19%	13.23%	14.29%
Total Applications	129,803	108,677	87,454

Notes: This table reports the fraction of applications per type of application. We report data for school years from 2019-20 to 2021-22.
Source: Italian Ministry of Education, Restricted Data.

Table 8: Teacher Characteristics

	2019-20	*2020-21*	*2021-22*
Family reasons			
Family Reunification	55.83%	57.16%	54.15%
Children's Age < 6	15.57%	15.12%	13.02%
Children's Age < 18	31.14%	32.37%	31.26%
Assistance to Family Member			
Child	1.12%	1.15%	1.19%
Spouse	0.28%	0.31%	0.31%
Parent	1.81%	2.00%	2.31%
Education and Certifications			
PhD	4.01%	3.78%	3.69%
Postgraduate specialization	4.92%	4.93%	5.07%
Advanced course (duration $>$ 1 year)	52.46%	53.77%	52.88%
Seniority			
Average years of tenured teaching	6.81	7.07	8.09
Over 25 years of tenured teaching	4.36%	11.12%	7.43%
Average years of untenured teaching (before tenure)	6.22	6.18	6.06
Special Priorities			
Blindness	0.01%	0.01%	0.01%
Haemodialysis	0.01%	0.01%	0.01%
Teachers moved by law in the last 8 years	3.19%	3.32%	2.84%
Long-term disability	2.11%	2.38%	2.40%
Provide care to family member	3.21%	3.46%	3.80%
Teachers moved by law in the last 8 years, asking to come back to the same municipality	3.28%	3.40%	2.92%
Spouse in the Army	0.37%	0.39%	0.37%
Public office in the local administration	0.22%	0.24%	0.19%
Trade union leave	0.02%	0.01%	0.01%

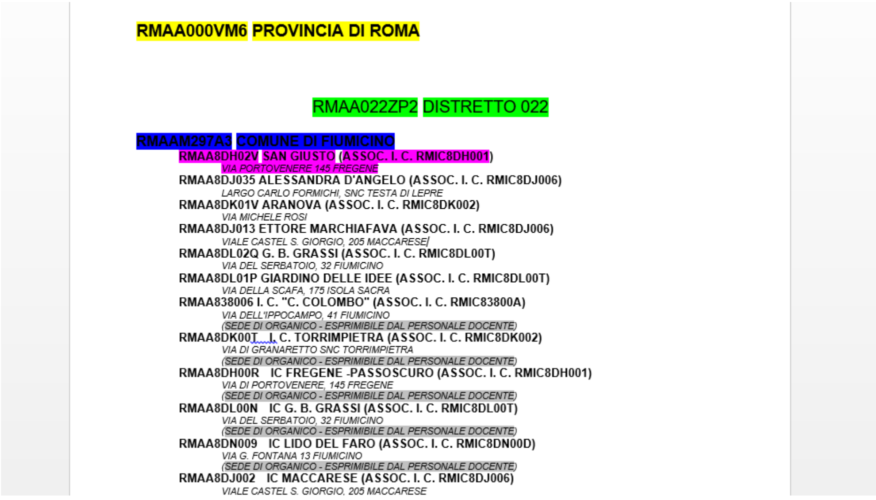
Notes: This table reports the main teacher characteristics, classified as: family reasons, education and qualifications, seniority, and special priorities. We report data for school years from 2019-20 to 2021-22.
Source: Italian Ministry of Education, Restricted Data.

Table 9: Schools in the Hierarchy

	N	Mean school buildings	Min	Max	N	Mean school-complexes	Min	Max
Panel A. Preschools								
Provinces	101	152.36	59	939	101	47.95	9	324
Districts	658	17.86	1	88	652	5.11	1	28
Small Districts	105	14.51	1	68	105	5.01	1	20
Municipalities	5,681	5.09	1	70	2,827	1.93	1	15
Big Municipalities	19	89.42	14	366	19	34.22	5	142
Schools	18,541				4,995			
Panel B. Primary schools								
Provinces	105	124.35	15	709	101	54.48	10	340
Districts	664	18.50	1	293	655	5.54	1	30
Small Districts	105	11.95	3	58	105	5.68	1	21
Municipalities	6,690	3.90	1	51	3,019	2.11	1	19
Big Municipalities	19	85.47	27	393	19	46.00	8	203
Schools	16,240				5,593			
Panel C. Middle schools								
Provinces	105	59.52	5	358	101	50.41	9	326
Districts	663	7.81	1	77	654	5.17	1	30
Small Districts	105	3.81	1	22	654	5.29	1	21
Municipalities	5,234	1.79	1	22	3,025	1.95	1	19
Big Municipalities	19	44.51	8	199	19	43.48	7	193
Schools	7,795				5,276			
Panel D. High schools								
Provinces	105	120.03	9	764	101	8,60	19	466
Districts	654	13.45	1	114	618	8.82	1	41
Small Districts	105	12.704	4	60	105	9.21	1	46
Municipalities	3,160	6.35	1	51	1,441	5.82	1	41
Big Municipalities	19	100.09	24	449	19	59.34	19	259
Schools	13,842				8,596			

Notes: This table reports information on the geographic hierarchical structure.
Source: Italian Ministry of Education, Restricted Data.

Figure 5: Official List of Schools (*Bollettini Ufficiali*)


RMAA000VM6 PROVINCIA DI ROMA

RMAA022ZP2 DISTRETTO 022

RMAAM297A3 COMUNE DI FIUMICINO
RMAA8DH02V SAN GIUSTO (ASSOC. I. C. RMIC8DH001)
VIA PORTOVENERE 145 FREGENE
RMAA8DJ035 ALESSANDRA D'ANGELO (ASSOC. I. C. RMIC8DJ006)
LARGO CARLO FORMICHI, SNC TESTA DI LEPRE
RMAA8DK01V ARANOVA (ASSOC. I. C. RMIC8DK002)
VIA MICHELE ROSI
RMAA8DJ013 ETTORE MARCHIAFAVA (ASSOC. I. C. RMIC8DJ006)
VIALE CASTEL S. GIORGIO, 205 MACCARESE
RMAA8DL02Q G. B. GRASSI (ASSOC. I. C. RMIC8DL00T)
VIA DEL SERBATOIO, 32 FIUMICINO
RMAA8DL01P GIARDINO DELLE IDEE (ASSOC. I. C. RMIC8DL00T)
VIA DELLA SCAFA, 175 ISOLA SACRA
RMAA838006 I. C. "C. COLOMBO" (ASSOC. I. C. RMIC83800A)
VIA DELL'IPPOCAMPO, 41 FIUMICINO
(SEDE DI ORGANICO - ESPRIMIBILE DAL PERSONALE DOCENTE)
RMAA8DK00T I. C. TORRIMPIETRA (ASSOC. I. C. RMIC8DK002)
VIA DI GRANARETTO SNC TORRIMPIETRA
(SEDE DI ORGANICO - ESPRIMIBILE DAL PERSONALE DOCENTE)
RMAA8DH00R IC FREGENE -PASSOSCURO (ASSOC. I. C. RMIC8DH001)
VIA DI PORTOVENERE, 145 FREGENE
(SEDE DI ORGANICO - ESPRIMIBILE DAL PERSONALE DOCENTE)
RMAA8DL00N IC G. B. GRASSI (ASSOC. I. C. RMIC8DL00T)
VIA DEL SERBATOIO, 32 FIUMICINO
(SEDE DI ORGANICO - ESPRIMIBILE DAL PERSONALE DOCENTE)
RMAA8DN009 IC LIDO DEL FARO (ASSOC. I. C. RMIC8DN00D)
VIA G. FONTANA 13 FIUMICINO
(SEDE DI ORGANICO - ESPRIMIBILE DAL PERSONALE DOCENTE)
RMAA8DJ002 IC MACCARESE (ASSOC. I. C. RMIC8DJ006)
VIALE CASTEL S. GIORGIO, 205 MACCARESE

Notes: This is the structure of the Official List of Schools (*Bollettini Ufficiali*), published by the Italian Ministry of Education. Highlighted in yellow, there is the province id and the name of the province (here, Rome). Highlighted in green, there is the district id, and the name of the district (here, District n.22). Highlighted in blue, there is the municipality id, and the name of the municipality (here, Fiumicino). Highlighted in pink, there is the school id, the name of the school, the id of the associated school complex, and the school address. Highlighted in grey, there is the specific reference for schools that can be listed in the teacher application (*Plesso sede di organico*).
Source: Italian Ministry of Education, Open Data.

ii. *Teacher rank order lists*. A dataset containing the rank order list of submitted preferences.

iii. *Teacher ownership rights*. A dataset containing information on the initial assignment of the teacher, namely the teaching position where they have their ownership right (*scuola o provincia di titolarità*).

iv. *Vacancies*. A dataset containing information about the vacancies available for new teaching positions.

v. *Assignment outcome*. A dataset containing the matching outcome.

vi. *Official Bullettin*. A dataset publicly available from the website of the Ministry of Education containing the Official List of the schools (*Bollettini Ufficiali*), from which we recover the exact order used in the tie-breaking procedure (Figure 5).

vii. *List of the schools*. A dataset containing the universe of the schools, with geographical information on the corresponding municipality, district and province. Since teachers can only indicate the school complex (*Plesso sede di organico*) in their applications,[40] we merge this dataset with the *Official Bullettin* to identify schools that can be listed.

Geographical Hierarchy

The geographical hierarchical structure that is relevant for teacher assignment has four levels: provinces, which include districts, which include municipalities, and which include schools.[41] Italy is currently divided into 20 administrative regions, which in turn contain 107 provinces, which in turn contain 7901 municipalities.[42] However, in the mobility of tenured teachers, administrative regions do not play any role, thus we will not consider them. Differently from the general administrative subdivision, the Ministry of Education considers also another level in the hierarchy, which is represented by the districts.[43] In total there are 769 districts and 6970 relevant municipalities (Table 9).[44]

Teacher applications

There were around 100,000 applications each year. Most of them, 40-41%, are from high school teachers, while the remaining 28-29% are from primary school teachers, 18% from middle school teachers, and 12-13% from preschool teachers (Table 4). The majority of the applications are requests for geographic transfers (82-83%).

Approximately 23-35% of the teachers are reallocated to their first stated position (See Figure 6 for the entire assignment rank distribution). Approximately 40-50% of the teachers fail to be reassigned to a more preferable position than their current positions. Most of the

[40]Teachers cannot list each specific institution, but only the school that is specifically designated as listable, within the entire school complex. See *Art. 9, Modalità di indicazione delle sedi di organico, Contratto Collettivo Nazionale Integrativo (2019/20, 2020/21, 2021/22)*.

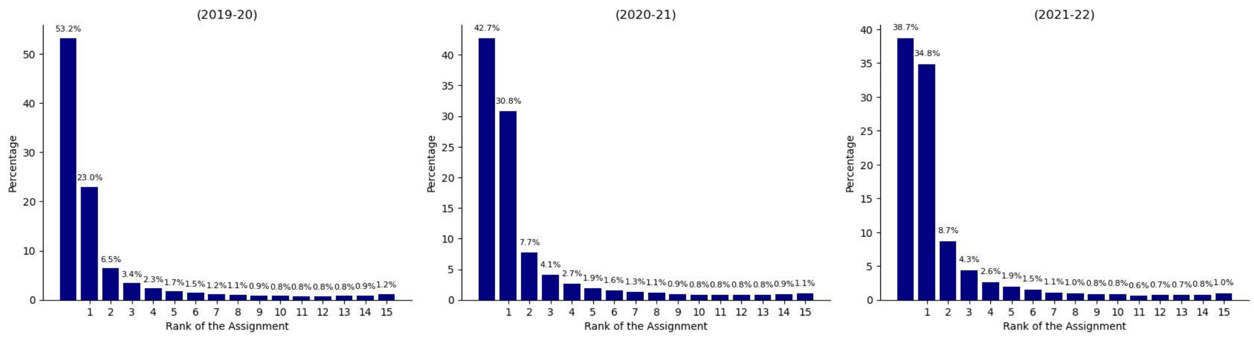
[41]In practice, there are big municipalities, not contained in any district, but containing small districts. In our analysis, big municipalities containing small district within them are Bari, Bologna, Cagliari, Catania, Firenze, Genova, Messina, Milano, Modena, Napoli, Padova, Palermo, Roma, Taranto, Torino, Treviso, Trieste, Venezia, Verona, Vicenza. However, the four level hierarchy is respected also in this case, by simply relabelling municipalities and districts.

[42]In this classification the provinces of Aosta, Fermo, Barletta-Trani-Andria and Sud Sardegna are not considered. At the same time the province of Bolzano is instead divided into three, according to the linguistic minorities: Bolzano (Italian language), Bolzano (Ladin language), Bolzano (German language). For simplicity we consider 101 provinces, excluding Trento, Bolzano (Italian language), Bolzano (Ladin language), Bolzano (German language).

[43]Districts were introduced in 1974 (art. 9 of DPR, n. 416, 31 May 1974) but along the years they have lost any effective role in the organization and administration of the educational network. However, they still play a role in the mobility of teachers, since teachers can submit this territorial entities as preferences.

[44]There can be municipalities without schools, thus this number is lower than the number of total municipalities.

Figure 6: Rank of the Assignment



Notes: These graphs show the position in the submitted rank ordered list for the assigned outcome. *Source*: Italian Ministry of Education, Restricted Data.

teachers tend to reapply across years (Table 10). Some teachers are not allowed to reapply because of a waiting-time constraint.[45]

Vacancies

The number of vacancies at each school is determined by four factors:

1. new teaching positions formed in that year,
2. positions that become vacant (e.g., because the previous teachers retire),
3. positions not assigned to permanent teachers, and
4. positions that become available during the assignment process itself (i.e., some teachers from that school participating in the reassignment system and securing a position in a different school)

School's priority orderings

We have constructed schools' priority orderings based on the available information.[46] Schools' priorities are determined roughly by five factors. First, teachers have the highest priority in their current school, such that if they cannot be transferred to another position, then they

[45] In the years we are considering, all teachers listing a school as a singleton school, who are successfully assigned to that school, are subject to a constraint to remain in that position for three years.

[46] Out of more than 40 different criteria that determine priorities (See Section B.1.5), we did not have access to few of those criteria and therefore we simplified the construction of schools' priorities. We expect that these criteria would only apply for a very limited number of teachers and would have negligible effect on the outcomes as far as our analysis is concerned.

Table 10: Reapplications

	2019-20	*2020-21*	*2021-22*
Applicants who apply in 2019-20, 2020-21, and *2021-22*	36,517 [31.61%]	36,517 [37.81%]	36,517 [46.68%]
Applicants who apply in 2019-20, and *2020-21*	60,329 [52.22%]	60,329 [62.47%]	-
Applicants who apply in 2019-20 and 2021-22	41,519 [35.94%]	-	41,519 [53.07%]
Applicants who apply in 2020-21, and *2021-22*	-	46,720 [48.38%]	46,720 [59.72%]
Applicants	115,534	96,577	78,232
Applications	129,803	108,677	87,454

Notes: This table reports the fraction of applicants reapplying multiple years. We report data for school years from 2019-20 to 2021-22.
Source: Italian Ministry of Education, Restricted Data.

are guaranteed to be reassigned to their current position (individual rationality constraint).[47] Second, there are "special priorities", for instance, due to specific health conditions and special circumstances.[48] Third, teachers gain priority depending on their "scores", which are calculated by considering seniority, educational and training qualifications, and family reasons.[49] Fourth, there are geographical priorities where a teacher gains priority in the municipality of her current school over applicants from other municipalities, and in the province of her current school over applicants from other provinces. The final priority ordering at each school is a strict ranking of teachers, where in case of ties after applying the above five factors, teacher age is used as a tie-breaker.

B.4 Simulations

We simulate an economy with 250 teachers and 280 schools. Each school has one vacant seat. There are 5 provinces, 35 districts and 140 municipalities. Within each municipality there are 2 schools. Each province contains 7 districts, and each district 4 municipalities.

[47] This feature is similar to some other assignment problems, such as the French teacher assignment (Combe et al., 2022a,b), the Danish day care assignment (Kennes et al., 2014, 2019), or on-campus housing in US (Guillen and Kesten, 2012).

[48] Appendix B.2.1 provides a complete list.

[49] See Appendix B.2 for a complete detailed list.

Table 11: Vacancies

	2019			2020			2021		
	Mean	Min	Max	Mean	Min	Max	Mean	Min	Max
Preschools									
Schools	1.6	0	32	2.3	0	36	2.4	0	36
Municipalities[a]	2.9	0	130	4.0	0	228	4.2	0	260
Districts	10.9	0	118	15.2	0	154	16.0	0	150
Provinces	80.8	2	528	111.9	12	866	118.1	10	918
Primary Schools									
Schools	4.1	0	58	6.1	0	66	7.8	0	70
Municipalities	8.2	0	800	12.3	0	1610	15.7	0	3004
Districts	32.7	0	284	49.3	0	334	62.8	0	504
Provinces	246.2	0	2596	368.2	36	3992	468.1	22	5320
Middle Schools									
Schools	8.6	0	84	10.2	0	90	13.6	0	104
Municipalities	16.2	0	1770	19.5	0	2232	26.1	0	2996
Districts	65.1	0	392	78.4	0	406	104.3	0	536
Provinces	489.5	52	3950	584.9	82	4676	776.9	110	6076
High Schools									
Schools	13.3	0	148	17.8	0	166	24.3	0	228
Municipalities	46.7	0	1098	63.1	0	1756	86.5	0	3078
Districts	69.2	0	500	94.9	0	596	130.5	0	808
Provinces	474.4	46	2632	647.2	78	3734	890.3	98	5390

Notes: The reported vacancies exclude the potential vacancies arising from positions that become available during the assignment process itself.
[a] Note that municipalities includes both small and big municipalities.
Source: Italian Ministry of Education, Restricted Data.

For each teacher t and each item i we assume that teacher preferences are derived from the following utility function:

$$U_t(i) = \underbrace{\rho \cdot V(i)}_{\text{common utility}} + \underbrace{(1-\rho) \cdot V_t(i)}_{\text{idiosyncratic utility}} \tag{1}$$

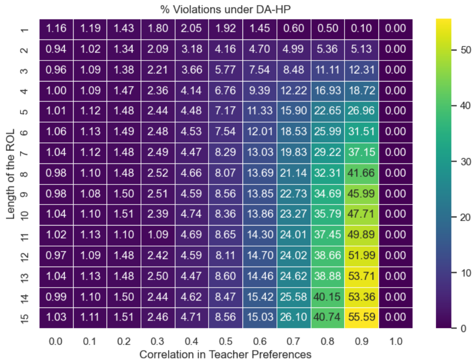
where ρ is a parameter that we make vary between 0 and 1, and $V(\cdot)$ are drawn from iid standard normal distributions, with mean 0 and variance 1. The first term of the utility reflects teachers' common evaluation of the item, and it only varies by item. The higher this term, the more correlated are teachers' preferences. The second term reflects an idiosyncratic preference, and varies between the pair of teacher-item.

For each school, priorities are determined following the current rules in the Italian teacher assignment. [50] This reflects the fact that in the Italian teacher assignment priority rules are common knowledge.

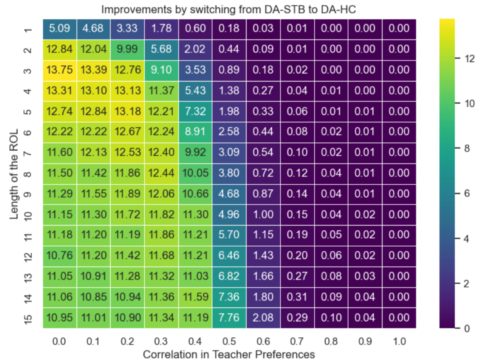
We run 1,000 simulations varying teachers' preference correlation parameter and the length of the ROL and report the results in Figure 7. The percentage of teachers with priority violations under DA-HP can be as high as more than half of the teachers and falls to zero when teacher preferences are perfectly aligned. In fact, in this case, DA-HP and DA-HC mechanisms coincide. The percentage of teachers improving by switching to DA-HC from the benchmark DA-STB can be as high as 14% of the teachers. The percentage decreases as the correlation increases, and falls to zero when preferences are perfectly aligned (i.e. when the two mechanisms coincide). Figures 7 also compares welfare improvements by switching to DA-HC from the current mechanism, DA-HP, and vice versa. Note that there is no Pareto dominance relation between the two mechanisms, though DA-HP would be generally more efficient. On the other hand, DA-HC eliminates JE and performs better in efficiency terms compared to the benchmark mechanisms that eliminates JE, the DA-STB.

[50]We randomly endow each teacher with a score and a school where they have ownership rights. From these elements we determine teachers' priorities at each school.

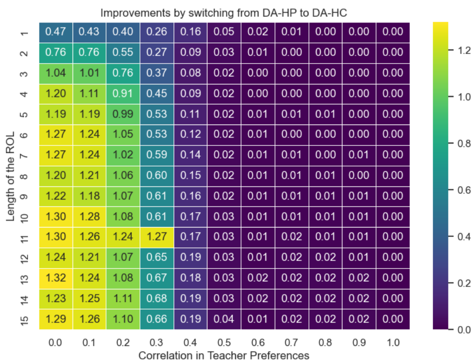
Figure 7: ROLs



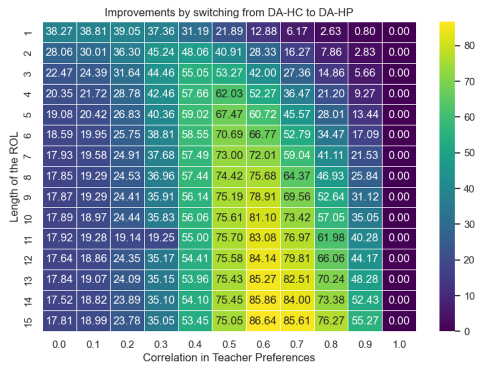
(a) Priority violations under DA-HP



(b) Welfare Improvements by Switching from DA-STB to DA-HC



(c) Welfare Improvements by Switching from DA-HP to DA-HC



(d) Welfare Improvements by Switching from DA-HC to DA-HP

Notes: Figure (a) shows the percentage of teachers whose priority rights are violated under DA-HP, by length of the ROL (y-axis) and correlation of teachers' preferences (x-axis). Figure (b) shows the percentage of teachers improving their welfare by switching from DA-STB to DA-HC, by length of the ROL (y-axis) and correlation of teachers' preferences (x-axis). Figure (c) shows the percentage of teachers improving their welfare by switching from DA-HP to DA-HC, by length of the ROL (y-axis) and correlation of teachers' preferences (x-axis). Figure (d) shows the percentage of teachers improving their welfare by switching from DA-HC to DA-HP, by length of the ROL (y-axis) and correlation of teachers' preferences (x-axis).