Matching with Weak Preferences: Redesigning Teacher Assignment in Italy*

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Abstract

Centralized assignment systems often require participants to submit strict rankings, even when they could rank some alternatives as tied. We show how mechanisms can elicit and use such ties to improve assignments. We introduce Deferred Acceptance with Coordinated Choice, a class of mechanisms that elicit weak preferences and attain the efficiency frontier of fair and strategy-proof mechanisms in a priority-based assignment model. We apply the framework to Italy's centralized teacher assignment system, where teachers' rank-order lists may include both individual schools and broader geographic regions representing ties among schools. We propose a redesign that eliminates priority violations and Pareto improves over the standard simple tie-breaking benchmark. Our theoretical results, combined with administrative data, show that ties can be a source of efficiency when properly incorporated into market design.

Keywords: Weak Preferences, Matching, Market Design, Teacher Assignment.

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1 Introduction

Ties in preferences are common in matching markets. In an online survey of university students in Singapore, 76% report that they would prefer to submit ties when ranking courses (Li, 2020). In Amsterdam's school choice system, administrative data combined with survey evidence show that some parents assign identical scores to multiple schools, even when encouraged to provide fine-grained rankings (De Haan et al., 2023). These ties create a design problem. Mechanisms that require strict rankings or resolve indifferences by simple tie-breaking may leave efficiency gains unrealized. Mechanisms that try to use weak preferences more directly, however, risk turning them into a source of priority violations or strategic incentives.

This tension is especially clear in Italy's centralized reassignment system for tenured public-school teachers.[1] Each year, approximately 100,000 teachers submit rank-order lists (ROLs) that may include both individual schools and broader geographic regions, with regions representing ties among schools.[2] Respecting teachers' priority rights is an explicit policy objective.[3] Yet the current mechanism, which seeks to accommodate weak preferences without simple tie-breaking, violates priorities and creates incentives for strategic misreporting. The resulting assignment disputes have generated extensive litigation and national-level scrutiny. In a 2016 parliamentary hearing, the Minister of Education referred to the algorithm used for the centralized reassignment of tenured teachers and reported that the system faced roughly 1,000–1,200 cases per year requiring formal conciliation.[4]

The policy failure does not lie in allowing teachers to rank geographic regions. Such reports are natural in a market where preferences are strongly shaped by geography. Most vacancies are concentrated in northern regions, while teachers' residence, origins, and applications are concentrated more heavily in southern regions.[5] This geographic imbalance helps explain why teachers often seek to move toward particular regions over the course of their careers. Regional entries reduce reporting costs and give the assignment system useful flexibility. The difficulty lies in using these ties without creating priority violations or strategic incentives. The current mechanism fails to meet this challenge.

[1] Teacher assignment is an important policy problem because teachers substantially affect students' educational and long-run outcomes (Araujo et al., 2016; Chetty et al., 2014; Rockoff, 2004).

[2] Similar practices exist for example in the intra-regional teacher assignment system in France (Silhol and Wilner, 2022; Terrier, 2014).

[3] This principle has been articulated in several judgments of the Italian courts as an expression of fundamental constitutional values, including non-discrimination and meritocracy. See, for instance, Tribunale di Salerno, Sezione Lavoro, Sentenza n. 336/2020, concerning a teacher who challenged his assignment after positions in a preferred territorial area were assigned to teachers in a later mobility phase and with lower scores.

[4] The transcript of the hearing is available on the website of the Italian Parliament: documenti.camera.it.

[5] In the 2016 parliamentary hearing, the Minister noted that 80% of teachers had residence or origins in central-southern Italy, while 65% of available teaching positions were located above the Rome line. See also Figure 3 for the geographic imbalance between applications and available positions.

This tension leads to our central question: can ties in submitted preferences be turned into efficiency gains without sacrificing fairness or strategy-proofness, or do these constraints force the designer back to simple tie-breaking?

We first study this question within the classical many-to-one, priority-based assignment model with outside options (Abdulkadiroğlu and Sönmez, 2003), extended to allow weak preferences. Agents on one side, whom we call teachers, may have weak preferences over schools and an outside option, while schools have capacities and strict priority orderings. The framework covers applications such as school choice, course allocation, daycare assignment, and entry-level teacher assignment, where priorities encode legal or procedural rights and agents may have weak preferences over alternatives. We then apply our findings to the centralized reassignment of tenured public-school teachers in Italy.

An assignment is *fair* if, whenever an agent prefers an institution to her assigned one, all positions at that institution are assigned to agents with higher priority (Abdulkadiroğlu and Sönmez, 2003; Balinski and Sönmez, 1999). Fairness—also called stability—is central in applications because priorities often reflect legal rights, and mechanisms that violate them are difficult to sustain in practice (Roth, 1991, 2002, 2008). A mechanism is *strategy-proof* for applicants if no agent can benefit from misreporting her preferences. Strategy-proofness makes truthful reporting safe by removing the need to anticipate what others will report. When this property fails, participants face a strategic burden that can disadvantage those who report honestly, while others who are sufficiently sophisticated, or simply fortunate, may manipulate the system successfully (Pathak and Sönmez, 2008).

The main question is what welfare gains for applicants remain attainable subject to fairness and strategy-proofness. Under strict preferences, the Deferred Acceptance (DA) mechanism is well known to achieve a strong welfare benchmark: it is fair and strategy-proof, and its outcome Pareto dominates every other fair assignment for applicants (Gale and Shapley, 1962). With weak preferences, however, we show that no strategy-proof mechanism can always select a *constrained Pareto efficient* assignment: a fair assignment that is not Pareto dominated by any other fair assignment.[6] This impossibility shifts the focus from undominated assignments to undominated mechanisms. Instead, we ask which fair and strategy-proof mechanisms are not Pareto dominated by any other fair and strategy-proof mechanism across all problems. We refer to such mechanisms as lying on the efficiency frontier of fair and strategy-proof mechanisms.[7]

[6] Indeed, even identifying a constrained Pareto efficient allocation is computationally intractable (NP-hard) under weak preferences (Manlove et al., 2002). By contrast, in settings *without priorities*—and thus without fairness constraints—Pareto efficiency and strategy-proofness are compatible (Alcalde-Unzu and Molis, 2011; Jaramillo and Manjunath, 2012; Saban and Sethuraman, 2013).

[7] Abdulkadiroğlu et al. (2009) earlier studied this efficiency frontier when ties arise in priorities rather than preferences and called mechanisms on the efficiency frontier as "undominated".

We introduce a class of mechanisms, Deferred Acceptance with Coordinated Choice (DA-CC), and show that every mechanism in this class lies on the efficiency frontier of fair and strategy-proof mechanisms (Theorem 1). We then show constructively that every Deferred Acceptance with Simple Tie-Breaking (DA-STB) mechanism is Pareto improved upon by some DA-CC mechanism (Theorem 2). Thus, DA-STB, the standard benchmark, does not lie on this frontier. This finding differs sharply from settings in which ties arise in priorities rather than preferences: there, DA-STB itself lies on the efficiency frontier of strategy-proof mechanisms (Abdulkadiroğlu et al., 2009).[8] Therefore, when ties arise in preferences, DA-CC provides a systematic way to improve upon simple tie-breaking in priority-based matching without sacrificing fairness or strategy-proofness.[9]

The key idea behind DA-CC is to preserve indifferences rather than resolve them prematurely. DA-STB converts an agent's indifferences into a strict order, so tied institutions are considered sequentially and independently. This loses information: the mechanism treats an agent who is willing to accept any institution in an indifference class as if she had strict preferences within that class. DA-CC instead treats each indifference class as a set of simultaneous applications: at each step, an agent applies to all institutions in her current indifference class, and a choice rule coordinates tentative acceptances across institutions. This coordination uses the flexibility in weak preferences while respecting priorities and preserving strategy-proofness.[10]

We next return to the Italian teacher assignment system to show how this general idea can be implemented in a concrete institutional setting. The current mechanism already tries to exploit the flexibility created by geographic ties through a variant of the DA algorithm, but it does so through local priority adjustments rather than coordinated choice. When a teacher ranks a region, schools in that region are considered sequentially according to an official ordering. If several teachers apply to the same school, the mechanism gives additional priority within the algorithm to applicants who listed the school more specifically, as long as broader applicants can still be considered for another vacancy in the same region. The intended logic is natural: a teacher who ranked a broad region may be accommodated elsewhere in the same indifference class. But the rule applies this logic myopically. A teacher whose claim is postponed because she ranked a broad region may later lose access to the entire region as competition changes during the algorithm. The mechanism therefore turns expressed indifference among schools into a weaker priority claim,

[8] Abdulkadiroğlu et al. (2009) strengthen an earlier result by Erdil and Ergin (2008), who show that any mechanism that always selects a constrained Pareto efficient assignment is not strategy-proof.

[9] Under more restrictive assumptions, earlier work in priority-based assignment shows that fair and strategy-proof benchmark mechanisms can be improved upon. For example, Li (2020) studies course allocation in Singapore with weak preferences and a single common priority ordering; Doğan and Raghavan (2024) analyze the COVAX vaccine allocation problem with dichotomous preferences and a single common priority ordering; and Aziz et al. (2025) study daycare assignment with dichotomous preferences and strict priorities.

[10] The DA-CC class includes mechanisms that can be implemented in polynomial time.

creating priority violations and incentives to replace regional entries with more specific ones.[11]

We take the official emphasis on priority rights and the current use of regional applications as pointing to a design goal: making better use of geographic flexibility while preserving fairness and strategy-proofness. We propose a specific DA-CC mechanism tailored to the Italian setting: Deferred Acceptance with Hierarchical Choice (DA-HC). DA-HC is designed for environments where indifferences arise from hierarchically nested geographic regions and fits the administrative structure of Italian teacher assignment. It preserves the existing reporting language and broad structure of the current procedure, replacing only the current mechanism's myopic rejection rule with a farsighted one. DA-HC is fair, strategy-proof, and Pareto improves upon the benchmark DA-STB. Moreover, DA-HC lies on the efficiency frontier of fair and strategy-proof mechanisms, subject to a mild invariance condition.[12] DA-HC therefore offers a targeted and readily implementable reform, addressing a documented policy failure within the existing institutional framework.[13]

Using rich administrative data from the Italian Ministry of Education for the teacher reassignment procedure in 2019–20, 2020–21, and 2021–22, we complement the theoretical results in two ways. First, we show that regional entries are empirically important. Nearly half of all ranked items correspond to regions rather than individual schools, and these entries are not confined to the bottom of lists: across years, 19–24% of first-ranked items and 27–34% of second-ranked items are regions. Second, we evaluate DA-HC directly on submitted ROLs. In the preschool-teacher subsample from 2019, which constitutes around 13% of all applications, 3.87% of teachers receive an assignment they prefer under DA-HC relative to DA-STB, while no teacher receives an assignment she prefers under DA-STB relative to DA-HC. This exercise is conservative in one respect: because the current mechanism gives teachers incentives to replace broad regional entries with more specific ones, submitted ROLs may understate the prevalence of weak preferences, which is precisely the channel through which DA-HC generates gains over DA-STB. At the same time, the current mechanism generates substantial priority violations: 1,310 teachers, corresponding to 12.9% of all teachers in the subsample, have their priority violated at least at one school. Under DA-HC, this number is zero, in line with theory. Additional simulations in a stylized priority-based assignment model show that the share of applicants who strictly prefer DA-CC to DA-STB can exceed 10%. These results demonstrate that it is possible to convert ties

[11] For example, on an online forum accessible at orizzontescuolaforum.net, a teacher wrote that, although they were willing to move to any municipality in the province of Catania to return to their family and were considering ranking the entire province, they had realized that they could increase their chances by instead listing specific municipalities or schools in Catania.

[12] In the Italian reassignment environment, teachers' existing positions are real school seats rather than outside options, so the frontier result from the classical setup does not immediately extend. The frontier statement is established relative to mechanisms satisfying the following invariance condition: adding a school unacceptable to all teachers does not affect the outcome.

[13] In this sense, our approach echoes the minimalist market design perspective of Sönmez (2023).

into meaningful assignment improvements without sacrificing fairness or strategy-proofness.[14]

Our application also contributes to the growing market-design literature on teacher assignment.[15] Within this literature, the closest papers are Combe et al. (2022) and Combe et al. (2025), who show how market design can improve efficiency and distributional outcomes in French teacher assignment. Their work studies mechanisms under strict preferences and focuses on objectives such as the equitable distribution of experienced teachers.[16] We complement this agenda by studying weak preferences in teacher assignment. Such preferences are especially relevant in teacher mobility systems, where geography is a central determinant of preferences and policymakers may want to let teachers express flexibility over sets of geographically related schools. We show that coordinated choice can use this flexibility to generate efficiency gains while preserving fairness and strategy-proofness.

The paper is organized as follows. Section 2 introduces the baseline model. Section 3 defines the design desiderata and the efficiency frontier. Section 4 presents the benchmark DA-STB mechanism. Section 5 introduces the DA-CC class and establishes its properties. Section 6 extends the framework to the Italian reassignment environment and introduces DA-HC. Section 7 provides administrative evidence on the gains from coordinated choice. Section 8 concludes. The Appendix contains proofs, and the Supplemental Appendix contains additional institutional and data details.

2 Model

We study a priority-based assignment model with weak preferences. We refer to the two sides of the market as *teachers* and *schools*. Let $T = \{t_1, \dots, t_{|T|}\}$ be a finite set of teachers and $S = \{s_1, \dots, s_{|S|}\}$ a finite set of schools.

Each school $s \in S$ has a capacity $q_s \in \mathbb{N}_+$ and a strict priority ordering $\succ_s$ over T, that is, a complete, transitive, and antisymmetric binary relation. Let $\succ = (\succ_s)_{s \in S}$ denote the priority profile.

Each teacher $t \in T$ has a weak preference relation R_t over $S \cup \{\emptyset\}$, that is, a complete and transitive binary relation, where $\emptyset$ denotes the outside option. Let $R = (R_t)_{t \in T}$ denote the

[14] Abdulkadiroğlu et al. (2009), using administrative data from the New York City high school match where ties arise in priorities rather than preferences, document substantial efficiency improvements over DA-STB without sacrificing fairness, but only at the expense of strategy-proofness.

[15] Several teacher labor markets are organized through centralized systems, including France (Combe et al., 2025, 2022; Silhol and Wilner, 2022; Terrier, 2014), German federal states (Klein and Baur, 2019), Portugal (Tomás, 2017), Turkey (Dur and Kesten, 2019), Teach for America in Chicago (Davis, forthcoming), Peru (Bobba et al., 2021; Ederer, 2023), Ecuador (Elacqua et al., 2022, 2021), Mexico (Pereyra, 2013), São Paulo (Elacqua and Rosa, 2025), Uruguay (Luschei and Carnoy, 2010), and Czech Republic and Slovakia (Cechlárová et al., 2016, 2015). A related empirical literature develops equilibrium models of teacher labor markets and shows how preferences over compensation and non-monetary amenities shape teacher sorting (Bates et al., 2025; Biasi et al., 2021; Bobba et al., 2021; Ederer, 2023; Laverde et al., 2026).

[16] See also Mishra et al. (2025), who study teacher transfer systems in India to improve distributional outcomes.

preference profile, and let P_t denote the strict preference relation associated with R_t. A school $s \in S$ is **acceptable** for teacher t if $s \; R_t \; \emptyset$. We assume that the outside option forms a singleton indifference class: for every teacher $t \in T$ and every school $s \in S$, either $s \; P_t \; \emptyset$ or $\emptyset \; P_t \; s$.

A problem is a tuple $(T, S, q, R, \succ)$. Throughout the baseline analysis, unless otherwise stated, T, S, q, and $\succ$ are fixed, and we identify a problem with its preference profile R.

A matching is an assignment of teachers to schools or to the outside option that respects capacity constraints. Formally, a **matching** is a correspondence $\mu : T \cup S \to T \cup S \cup \{\emptyset\}$ such that, for each teacher $t \in T$ and each school $s \in S$: (i) $\mu(t) \in S \cup \{\emptyset\}$; (ii) $\mu(s) \subseteq T$; (iii) $\mu(t) = s$ if and only if $t \in \mu(s)$; and (iv) $|\mu(s)| \leq q_s$.

A **mechanism** elicits preferences from teachers and produces a matching. Given a mechanism φ and a preference profile R, we denote the assignment of teacher $t \in T$ by $\varphi_t(R)$.

3 Design Objectives

3.1 Fairness and Incentives

In priority-based assignment problems, fairness is typically defined as the combination of three conditions: *individual rationality*, *non-wastefulness*, and the *elimination of justified envy*.

Given a problem R, a matching μ is *individually rational* if each teacher is assigned an acceptable school or the outside option. That is, for each $t \in T$, $\mu(t) \; R_t \; \emptyset$.

A matching μ is *non-wasteful* if no teacher prefers a school with available capacity to her assignment. That is, there is no $t \in T$ and $s \in S$ such that $s \; P_t \; \mu(t)$ and $|\mu(s)| < q_s$.

A matching μ *eliminates justified envy* if, whenever a teacher $t \in T$ prefers the assignment of another teacher $t' \in T$ to her own assignment, teacher t' has higher priority than t at that school. That is, if $\mu(t') \; P_t \; \mu(t)$, then $t' \succ_{\mu(t')} t$.

A matching is **fair** (also known as *stable*) if it is *individually rational*, *non-wasteful*, and *eliminates justified envy*.

We say that a mechanism satisfies a matching property if, for every problem, the matching it selects satisfies that property. Thus, for example, a mechanism is fair if it selects a fair matching at every problem.

Beyond fairness, a desirable mechanism should also make it safe for participants to report their true preferences. A mechanism φ is **strategy-proof** if, for each preference profile R, no teacher $t \in T$ can benefit by misreporting her preferences. That is, for any R'_t, we have $\varphi_t(R) \; R_t \; \varphi_t(R'_t, R_{-t})$.

3.2 Efficiency and the Frontier

Efficiency is a fundamental objective in assignment problems, typically considered alongside fairness and strategy-proofness. However, as we show in Propositions 1 and 2, standard efficiency notions fall short under weak preferences when fairness constraints are imposed and strategy-proofness is required.

A natural starting point is the classical notion of Pareto efficiency. Given a problem R, a matching μ *Pareto dominates* another matching μ' if each teacher weakly prefers μ to μ', and at least one teacher strictly prefers μ to μ'. A matching μ is *Pareto efficient* if there is no matching μ' that Pareto dominates μ. A matching is **constrained Pareto efficient** if it is *fair* and not Pareto dominated by any other *fair* matching.

Another intuitive criterion considers the number of teachers assigned to acceptable schools. Given a problem R, a matching μ *size dominates* another matching μ' if μ assigns more teachers to acceptable schools than μ', that is,

$$|\{t \in T : \mu(t) \in S \text{ and } \mu(t)\ R_t\ \emptyset\}| > |\{t \in T : \mu'(t) \in S \text{ and } \mu'(t)\ R_t\ \emptyset\}|.$$

A matching μ is *size efficient* if there is no matching μ' that size dominates it. A matching μ is **constrained size efficient** if it is *fair* and there is no *fair* matching μ' that assigns more teachers to schools strictly preferred to the outside option.

Despite their appeal, we show that neither *constrained Pareto efficiency* nor *constrained size efficiency* is compatible with *strategy-proofness*.[17]

Proposition 1. *There is no mechanism that is strategy-proof and constrained Pareto efficient.*

Proposition 2. *There is no mechanism that is strategy-proof and constrained size efficient.*

Proofs are in Appendices A.1 and A.2, respectively. These impossibility results shift the relevant efficiency question from assignments to mechanisms. We say that a mechanism φ' **Pareto improves over** φ if (i) for each problem R, either $\varphi'(R) = \varphi(R)$ or $\varphi'(R)$ Pareto dominates $\varphi(R)$, and (ii) there exists a problem R' such that $\varphi'(R')$ Pareto dominates $\varphi(R')$.

We say that a fair and strategy-proof mechanism φ lies on the **efficiency frontier** of fair and strategy-proof mechanisms if there is no fair and strategy-proof mechanism φ' that Pareto improves over φ.

[17] Erdil and Ergin (2017) similarly show that there is no mechanism that is *strategy-proof* and *constrained Pareto efficient* when indifferences are allowed on both sides: teachers' preferences and schools' priorities. However, their result crucially relies on indifferences in priorities and therefore does not imply our impossibility result.

Frontier optimality is a mechanism-level efficiency requirement. To verify it, we introduce a problem-level sufficient condition. A matching μ is **constrained Pareto-size efficient** if it is *fair* and there exists no other *fair* matching μ' that both Pareto dominates and size dominates μ.

This condition is weaker than both *constrained Pareto efficiency* and *constrained size efficiency*, but it is strong enough to certify frontier optimality.

Proposition 3. *Every strategy-proof and constrained Pareto-size efficient mechanism lies on the efficiency frontier of fair and strategy-proof mechanisms.*

The proof is in Appendix A.3. Proposition 3 also implies that, among fair and strategy-proof mechanisms, any mechanism that Pareto improves another must also size improve it at some problem.[18] This does not mean, however, that every problem at which a Pareto improvement occurs must also involve a size improvement.

4 Benchmark: Deferred Acceptance with Simple Tie-Breaking

A natural benchmark resolves each teacher's indifferences by a fixed exogenous ordering of schools and then applies deferred acceptance. Fix a strict ordering $\triangleright$ on S. For each teacher t, let $R_t^{\triangleright}$ be the strict preference relation obtained from R_t by preserving the order of indifference classes in R_t and ranking schools within each indifference class according to $\triangleright$. The outside option keeps its position in the preference order.

The **Deferred Acceptance with Simple Tie-Breaking** mechanism, denoted DA-STB, is defined as follows.

DA-STB Algorithm

Step 1. Each teacher applies to her most preferred acceptable school according to $R_t^{\triangleright}$. Each school tentatively accepts its highest-priority applicants up to capacity and rejects all other applicants.

Step $k \geq 2$. Each teacher rejected in Step $k-1$ applies to her most preferred acceptable school according to $R_t^{\triangleright}$ that has not previously rejected her. Each school considers its tentatively accepted teachers together with its new applicants, tentatively accepts its highest-priority applicants up to capacity, and rejects all others.

The algorithm terminates when no rejection occurs. Teachers who are not tentatively accepted by any school are assigned to the outside option. The final assignment is the DA-STB outcome.

[18]This implication is reminiscent of size-improvement results in Erdil (2014), Alva and Manjunath (2020), and Afacan and Dur (2023).

When teachers' preferences are strict, DA-STB coincides with the classical deferred acceptance mechanism and is both *fair* and *strategy-proof* (Abdulkadiroğlu and Sönmez, 2003). With weak preferences, however, DA-STB resolves indifferences before the assignment process begins. This premature tie-breaking discards useful flexibility: a teacher who is indifferent among several schools is treated as if she strictly prefers one of them. As Theorem 2 shows, this loss is substantive: DA-STB does not lie on the efficiency frontier of fair and strategy-proof mechanisms. This contrasts with settings in which indifferences arise in school priorities rather than teacher preferences. In those settings, no *strategy-proof* mechanism can Pareto dominate DA-STB (Abdulkadiroğlu et al., 2009).

5 Deferred Acceptance with Coordinated Choice

DA-STB resolves each indifference class into a strict order before deferred acceptance begins. Our alternative keeps each indifference class intact. At each step, a teacher applies simultaneously to all schools in her current acceptable indifference class, and a choice rule coordinates tentative acceptances across schools.

An **application profile** is a collection

$$A = \{(t, A(t)) : t \in T\},$$

where $A(t) \subseteq S$ is the set of schools to which teacher t applies. The set $A(t)$ may be empty. A **choice rule** C assigns to each application profile A a matching $C(A)$ such that each assigned teacher is matched to a school in her application set: for each $t \in T$, either $C_t(A) \in A(t)$ or $C_t(A) = \emptyset$.

We impose three properties on choice rules. First, C is **maximum-size** if, for every application profile A, $C(A)$ assigns the largest possible number of teachers among all matchings that respect A. Second, C is **priority-respecting** if, whenever a teacher t is unassigned, every school to which she applied is filled only with teachers who have higher priority than t. Formally, if $C_t(A) = \emptyset$, then for every $s \in A(t)$ and every $t' \in C_s(A)$, we have $t' \succ_s t$. Third, C is **substitutable** if removing one teacher's application never causes another previously assigned teacher to become unassigned. Formally, for every A, every $t \in T$ with $C_t(A) \neq \emptyset$, and every $t' \neq t$, we have $C_t(A_{-t'}) \neq \emptyset$, where $A_{-t'}$ is obtained from A by replacing $A(t')$ with $\emptyset$.

Given any choice rule C, the **Deferred Acceptance with Coordinated Choice** mechanism induced by C, denoted DA-CC(C), is defined as follows. In the results below, we focus on DA-CC mechanisms induced by choice rules satisfying maximum-size, priority-respecting, and substitutability.

DA-CC Algorithm

Step 1. Each teacher applies to her most preferred acceptable indifference class. Teachers with no acceptable school do not apply. Let A^1 be the resulting application profile. Each teacher tentatively assigned by $C(A^1)$ is accepted at that school. Each teacher not assigned by $C(A^1)$ is rejected by all schools in her application set.

Step $k \geq 2$. Each teacher rejected in Step $k-1$ applies to her next most preferred acceptable indifference class that she has not previously applied to, if one exists; otherwise, she makes no application. Teachers not rejected in Step $k-1$ keep their previous application sets. Let A^k be the resulting application profile. Each teacher tentatively assigned by $C(A^k)$ is accepted at that school. Each teacher not assigned by $C(A^k)$ is rejected by all schools in her current application set.

The algorithm terminates when no rejection occurs. Teachers who are not tentatively assigned to any school are assigned to the outside option. The final assignment is the DA-CC outcome.

The three conditions on C are sufficient for DA-CC to preserve the standard incentive and fairness guarantees while attaining the relevant efficiency frontier.

Theorem 1. *For any maximum-size, priority-respecting, and substitutable choice rule C, the DA-CC mechanism induced by C is fair, strategy-proof, and lies on the efficiency frontier of fair and strategy-proof mechanisms.*

The proof is in Appendix A.5. Coordinated choice parallels the role of choice functions in the matching-with-contracts framework of Hatfield and Milgrom (2005): one can view the collection of schools as a single institution, and each school as a contractual term specifying where a teacher may be assigned. However, the standard framework does not apply directly because agents there have strict preferences over contracts, whereas teachers here may be indifferent among several schools. Thus a teacher's proposal is not naturally a single contract: she may apply to an entire indifference class at once, while the choice rule determines both whether she is accepted and, if so, at which school in that class. The key idea is therefore to relate DA-CC to a cumulative-offer procedure in an auxiliary market following Hatfield and Kominers (2019). Frontier optimality is shown by establishing constrained Pareto-size efficiency.

We next show constructively that the DA-CC class is nonempty and contains mechanisms whose outcomes can be computed efficiently. Fix a strict ordering $\rhd$ on S. For an application profile A, consider all teacher-school pairs (t, s) such that $s \in A(t)$. Order these pairs lexicographically:

(t,s) precedes (t',s') if either $s \rhd s'$, or $s = s'$ and $t \succ_s t'$. Let $x_1 \vdash x_2 \vdash \cdots \vdash x_m$ denote this ordering.

The choice rule $C^{\rhd}$ is defined as follows. Given A, first compute $L(A)$, the maximum number of teachers that can be assigned subject to capacities and application constraints. Then process the pairs in $\vdash$-order. Include pair $x_k = (t,s)$ if and only if there exists a matching respecting A of size $L(A)$ that includes all previously included pairs and also includes (t,s). Otherwise, exclude x_k. The rule returns the matching consisting of the included pairs.

Proposition 4. *For each ordering $\rhd$ over schools, the choice rule $C^{\rhd}$ is maximum-size, priority-respecting, and substitutable.*

The proof is in Appendix A.4. Therefore, $C^{\rhd}$ gives rise to a DA-CC mechanism satisfying Theorem 1.[19]

The following simple example illustrates the workings of DA-CC and the source of its improvement over DA-STB. DA-CC keeps each indifference class intact and coordinates assignments within it, whereas DA-STB first resolves indifferences according to the fixed order and then runs deferred acceptance.

Example 1. Let $T = \{t_1, t_2\}$ and $S = \{s_1, s_2\}$, with $q_{s_1} = q_{s_2} = 1$. Suppose that $s_1 \rhd s_2$ and that priorities and preferences are as follows:

R_{t_1}	R_{t_2}	$\succ_{s_1}$	$\succ_{s_2}$
$\{s_1, s_2\}$	s_1	t_1	t_1
$\emptyset$	$\emptyset$	t_2	t_2

Under DA-STB, teacher t_1 applies first to s_1, because $s_1 \rhd s_2$, and is accepted. Teacher t_2 also applies to s_1 and is rejected. Hence the DA-STB outcome assigns t_1 to s_1 and t_2 to the outside option.

Under the DA-CC mechanism induced by $C^{\rhd}$, teacher t_1 applies to the set $\{s_1, s_2\}$, while teacher t_2 applies to s_1. The maximum possible number of assignments is two, and the choice rule assigns t_2 to s_1 and t_1 to s_2. Thus DA-CC makes t_2 strictly better off and leaves t_1 indifferent. Therefore DA-CC Pareto dominates DA-STB at this problem.

The next theorem shows that this example reflects a general dominance relation.

Theorem 2. *The DA-STB mechanism based on $\rhd$ is Pareto dominated by the DA-CC mechanism induced by $C^{\rhd}$.*

[19] The induced mechanism is polynomial-time implementable: computing $L(A)$ and the feasibility tests defining $C^{\rhd}$ reduce to standard max-flow computations.

The proof is in Appendix A.6. The DA-CC framework offers flexibility because any choice rule satisfying the three properties yields the desired guarantees. This allows the choice rule to be tailored to the institutional environment. In the next section, we introduce a specific choice rule for settings where indifferences are generated by a geographic hierarchy, so that the resulting mechanism fits naturally within the institutional structure of Italian teacher assignment.

6 Application: Teacher Assignment in Italy

We now apply the design insights from the baseline model to Italy's centralized teacher reassignment system. The Italian environment differs from the baseline model in one important respect: teachers already hold school positions, and retaining an existing position consumes a real school seat rather than an outside option. At the same time, the institutional rules have the same priority-based logic as the baseline model: schools rank teachers by administrative priorities, teachers are protected at their current schools through priority rights, and the assignment procedure is a variant of deferred acceptance. We incorporate the reassignment feature by treating each teacher's current position as an endowment school. The design problem is then to reassign teachers to weakly preferred schools while preserving priority rights and using the flexibility created by regional applications.

6.1 Institutional Setting

Each year, roughly 100,000 tenured teachers apply for reassignment through a national mobility procedure. Teachers submit rank-order lists of schools and geographic regions, while schools rank teachers according to strict administrative priority orderings, which combine teacher scores, special legal priorities, and geographical priority rights tied to teachers' current schools.[20] The Ministry then assigns teachers to available positions through a centralized algorithm.

Schools are organized in a nested geographic hierarchy: schools belong to municipalities, municipalities to districts, and districts to provinces.[21] Let $\mathcal{G}$ denote the set of geographic regions. Each $G \in \mathcal{G}$ is a nonempty subset of schools, $\bigcup_{G\in\mathcal{G}} G = S$, and any two regions are either disjoint or nested.

Each teacher submits an ROL of at most 15 items. An item may be either a school or a region. Ranking a region is interpreted as ranking all schools in that region, except for schools already included in a higher-ranked item. Thus, if teacher t ranks school s first, a region G containing s second, and school $s' \notin G$ third, then the induced preference relation treats s as strictly above every other school in G, all schools in $G \setminus \{s\}$ as tied, and each of them as strictly above s'.

[20] Appendix B.1 describes the priority criteria. Priorities are strict because teacher age is eventually used as a tie-breaker.

[21] Some metropolitan cities contain subregions that are also referred to as districts. Appendix B.1 provides details.

Formally, for any school s and ROL R_t, let $rank(s, R_t)$ be the highest-ranked item in R_t that contains s. Teacher t weakly prefers s to s' if $rank(s, R_t) \leq rank(s', R_t)$, strictly prefers s to s' if the inequality is strict, and is indifferent between them if equality holds. In particular, if a ranked region contains a teacher's current school, then the current school is tied with the other schools in that region, except for schools already included in a higher-ranked item.

Let $\omega_t \in S$ denote teacher t's current school, and let $\omega = (\omega_t)_{t \in T}$ denote the endowment profile. In the reassignment problem, a school s is acceptable for teacher t if $s \; R_t \; \omega_t$. We assume that the endowment profile is feasible: $|\{t \in T : \omega_t = s\}| \leq q_s$ for each $s \in S$. We also assume that each teacher t is ranked among the top q_{ω_t} teachers at her current school ω_t. Thus, if a teacher cannot be reassigned to a weakly preferred school, she can retain her current position.

Thus, the Italian reporting language induces weak preferences with a hierarchical structure. This is the subclass of weak preferences to which we apply the DA-CC framework.

6.2 The Current Mechanism

The Ministry currently assigns teachers using a variant of deferred acceptance.[22] When a teacher's current ROL item is a region, the algorithm considers the schools in that region sequentially according to an official ordering.[23] We assume that school indices follow this ordering.

The distinctive feature is how the mechanism treats regional applications. When a teacher applies through a regional item, the mechanism has the teacher apply to the schools in that region sequentially, according to the official order. At a school, applicants who listed the school more specifically are considered before applicants who listed it through a coarser region, provided that the coarser region still contains another school with vacancy at that step, meaning that the school has received fewer applications than its capacity up to that step.[24] Thus, a teacher may lose priority at one school because the mechanism treats another school in the same ranked region as a possible substitute. The intended rationale is natural: a teacher who ranked a broader region may still be accommodated elsewhere within the same indifference class. The problem is that this logic is applied myopically.

We refer to this procedure as *Deferred Acceptance with Hierarchical Priorities* (DA-HP). Appendix B.1.1 gives the formal algorithm. The following example shows that DA-HP is neither fair nor strategy-proof.

[22]The assignment rules are described in the national collective bargaining agreement, *Contratto Collettivo Nazionale Integrativo concernente la mobilità del personale docente, educativo ed A.T.A*, and in the ministerial decree on teacher mobility, *Ordinanza sulla mobilità del personale docente, educativo ed ATA*.

[23]This ordering is published annually by the Ministry of Education in the *Bollettini Ufficiali*.

[24]This rule is set out in Article 6, paragraph 5, of the national collective bargaining agreement.

Example 2. Let $T = \{t_1, t_2, t_3, t_4\}$ and $S = \{s_1, s_2, s_3, s_4\}$. Let $r_1 = \{s_1, s_2\}$, $r_2 = \{s_3, s_4\}$, and $\phi = r_1 \cup r_2$. Let $q_{s_1} = q_{s_2} = q_{s_3} = 1$, $q_{s_4} = 4$, and $\omega_t = s_4$ for each $t \in T$. ROLs and priorities are:

R_{t_1}	R_{t_2}	R_{t_3}	R_{t_4}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
r_1	s_1	s_3	s_3	t_1	t_3	t_4	t_1
s_4	s_4	s_2	s_4	t_3	t_2	t_3	t_2
		s_4		t_2	t_1	t_1	t_3
				t_4	t_4	t_2	t_4

Schools omitted from a teacher's column are ranked below her current school. In the first step of DA-HP, t_1 and t_2 apply to s_1, while t_3 and t_4 apply to s_3. Although $t_1 \succ_{s_1} t_2$, teacher t_1 is rejected and t_2 is tentatively accepted at s_1, because s_2, another school in t_1's ranked region r_1, has vacancy. In the second step, t_1 applies to s_2 but is rejected because t_3 applies to s_2 and $t_3 \succ_{s_2} t_1$.

Thus, DA-HP is not fair: t_1 loses s_1 to a lower-priority teacher. It is also not strategy-proof: t_1 can obtain s_1 by reporting only s_1 as acceptable.

6.3 Deferred Acceptance with Hierarchical Choice

We now construct a DA-CC mechanism tailored to the hierarchical structure of Italian teacher assignment. Because teachers may apply either to a school or to a geographic region, the relevant application profiles have a special form.

A *hierarchical application profile* is an application profile $A : T \to 2^S$ such that, for each teacher $t \in T$,

$$A(t) \in \{\emptyset\} \cup \{\{s\} : s \in S\} \cup \mathcal{G}.$$

Thus, each teacher applies to either no school, one school, or all schools in a geographic region. For any teacher t, let A_{-t} denote the application profile obtained from A by replacing $A(t)$ with $\emptyset$ and leaving all other applications unchanged. For convenience, write (t, s) instead of $(t, \{s\})$ for singleton applications.

We say that teacher t has a **finest application** in A if $A(t) \neq \emptyset$ and there is no teacher $t' \in T$ with $A(t') \neq \emptyset$ such that $A(t') \subsetneq A(t)$. Among teachers with finest applications, the **smallest-index teacher among finest applications** is the one with the smallest index.

A teacher t is **critical** in A if removing her application reduces the maximum number of teachers that can be assigned. Formally, t is critical in A if the maximum size of a matching that respects A is strictly greater than the maximum size of a matching that respects A_{-t}. Equivalently,

t is critical if she is assigned in every maximum-size matching that respects A.

Example 3. Let us reconsider the problem in Example 2 and the application profile $A = \{(t_1, r_1), (t_2, s_1), (t_3, s_3), (t_4, s_3)\}$. The maximum number of teachers that can be assigned is 3: for example, assign t_1 to s_2, t_2 to s_1, and t_3 to s_3. Removing either t_1's or t_2's application lowers the maximum size to 2, so both are critical. By contrast, removing t_3 or t_4 does not reduce the maximum size. Hence, only t_1 and t_2 are critical in A.

We now define the *Hierarchical Choice* rule, denoted by HC. The rule processes applications from finer to coarser sets. The key distinction is between applicants who are critical for maintaining the maximum number of assignments and applicants who are not. A critical applicant is assigned to a vacant feasible school; a noncritical applicant is processed through the usual priority-based rejection logic.

HC ***Algorithm***

Fix a hierarchical application profile A. Throughout, consider only teachers with nonempty applications.

Step 1. Consider the smallest-index teacher among finest applications, say t. Tentatively assign t to the smallest-index school in $A(t)$.

Step $k \geq 2$. Among teachers who are not tentatively assigned and have not been rejected by all schools in their applications, consider the smallest-index teacher among finest applications, say t. If no such teacher exists, stop.

<u>Case 1.</u> If t is critical in A, tentatively assign t to the smallest-index school in $A(t)$ with a vacant seat.[25]

<u>Case 2.</u> If t is not critical in A, let s be the smallest-index school in $A(t)$ that has not rejected t. If s has a vacant seat, tentatively assign t to s. If s has no vacant seat but t has higher priority than the lowest-priority teacher tentatively assigned to s, tentatively assign t to s and reject that teacher. Otherwise, reject t from s.

The HC algorithm is a deferred-acceptance-type procedure applied to an application profile rather than to a full preference profile. Teachers are considered one at a time, with finer applications taking precedence. Critical teachers are assigned to vacant seats; noncritical teachers are processed as in a standard priority-based rejection procedure. This is the key difference from

[25] By Lemma 4, such a school exists.

DA-HP: a regional applicant is asked to wait only when doing so is safe for preserving maximum size.

Example 4. Let us reconsider $A = \{(t_1, r_1), (t_2, s_1), (t_3, s_3), (t_4, s_3)\}$ from Example 3. The finest applications are (t_2, s_1), (t_3, s_3), and (t_4, s_3), so t_2 is considered first and tentatively assigned to s_1. Next, t_3 is considered and tentatively assigned to s_3. Then t_4 is considered; since $t_4 \succ_{s_3} t_3$, she replaces t_3 at s_3. Finally, t_1 is considered. Since t_1 is critical, she is assigned to the smallest-index school in $A(t_1)$ with a vacant seat, namely s_2. Hence, $HC(A) = \{(t_1, s_2), (t_2, s_1), (t_4, s_3)\}$.

We now define **Deferred Acceptance with Hierarchical Choice** (DA-HC). The mechanism is the reassignment analogue of DA-CC induced by the hierarchical choice rule HC: it has the same deferred-acceptance structure and uses HC to process applications at each step. The difference is that a teacher's current school is not an outside option. It is a real school seat and may be reached as an application; if reached, it is processed by HC together with all other applications.

DA-HC Algorithm

Step 1. Each teacher applies to her most preferred ROL item that contains at least one school weakly preferred to her current school. Let A^1 be the resulting hierarchical application profile. The choice rule HC is applied to A^1. Teachers assigned by $HC(A^1)$ are tentatively accepted at the corresponding schools; teachers not assigned by $HC(A^1)$ are rejected from their current application item.

Step $k \geq 2$. Each teacher rejected in Step $k-1$ applies to her next most preferred ROL item that contains at least one school weakly preferred to her current school and that she has not previously applied to, if one exists. Teachers not rejected in Step $k-1$ keep their previous application items. Let A^k be the resulting hierarchical application profile. The choice rule HC is applied to A^k. Teachers assigned by $HC(A^k)$ are tentatively accepted at the corresponding schools; teachers not assigned by $HC(A^k)$ are rejected from their current application item.

The algorithm terminates when no rejection occurs. The final assignment is the DA-HC outcome.

Because each teacher has sufficiently high priority at her current school, a teacher who reaches her current school cannot ultimately be forced below it. We call the DA-CC mechanism induced by HC the **Deferred Acceptance with Hierarchical Choice** mechanism, or DA-HC. The next example illustrates the mechanism.

Example 5. Let us reconsider the problem in Example 2. In Step 1 of DA-HC, each teacher applies to her top-ranked item, so the application profile is $A = \{(t_1, r_1), (t_2, s_1), (t_3, s_3), (t_4, s_3)\}$. By Example 4, $HC(A) = \{(t_1, s_2), (t_2, s_1), (t_4, s_3)\}$, so t_3 is rejected. In Step 2, t_3 applies to her next-best acceptable item, s_2. The new application profile is $A' = \{(t_1, r_1), (t_2, s_1), (t_3, s_2), (t_4, s_3)\}$. Under HC, $HC(A') = \{(t_1, s_1), (t_3, s_2), (t_4, s_3)\}$, so t_2 is rejected. In Step 3, t_2 applies to her current school s_4 and is accepted. The final DA-HC outcome is $\{(t_1, s_1), (t_2, s_4), (t_3, s_2), (t_4, s_3)\}$.

Theorem 3. *DA-HC is fair, strategy-proof, and Pareto improves over DA-STB.*

The proof is in Appendix A.7. The key step is to show that HC is maximum-size, priority-respecting, and substitutable. These properties concern the choice rule on application profiles, and are therefore unaffected by whether participation is modeled through an outside option or through endowment schools. Fairness and strategy-proofness then follow from Theorem 1: any violation in the reassignment environment would induce a corresponding violation in the baseline outside-option model. The dominance comparison with DA-STB is proved directly for the reassignment environment. The following example shows that the Pareto improvement over DA-STB can also increase size.

Example 6. Let $T = \{t_1, t_2\}$ and $S = \{s_1, s_2, s_3\}$. Let $r_1 = \{s_1, s_2\}$, $r_2 = \{s_3\}$, and $\phi = r_1 \cup r_2$. Let $q_{s_1} = q_{s_2} = 1$, $q_{s_3} = 2$, and $\omega_t = s_3$ for each $t \in T$. ROLs and priorities are:

R_{t_1}	R_{t_2}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
r_1	s_1	t_1	t_1	t_1
		t_2	t_2	t_2

Under DA-STB, t_1 is assigned to s_1 and t_2 remains at her current school s_3. Under DA-HC, t_1 is assigned to s_2 and t_2 is assigned to s_1. Thus, DA-HC Pareto improves over DA-STB by making t_2 better off while leaving t_1 indifferent. It also size improves over DA-STB by increasing the number of teachers assigned to schools they prefer to their current schools from 1 to 2.

The frontier result in the baseline model does not apply directly because, in the reassignment environment, a teacher's fallback is a real school seat rather than an outside option. We establish an analogous frontier result within the class of mechanisms satisfying a mild invariance property. Assume that each problem can be enlarged by adding a school, with some positive capacity and some priority ordering, that every teacher finds worse than her endowment.[26] We require that such an addition leave the mechanism's assignment of teachers among the original schools

[26] This does not convert the reassignment problem into an outside-option problem: the added school need not have enough capacity to accommodate all teachers, and the original scarcity of school seats is preserved.

unchanged. DA-HC satisfies this property because the added school is never reached by any teacher in the application process, and hence never enters any application profile processed by HC.

Corollary 1. *DA-HC lies on the efficiency frontier of fair and strategy-proof mechanisms that satisfy the invariance property above.*

The proof is in Appendix A.8. The result follows from Theorem 1, which places every strategy-proof and constrained Pareto-size efficient DA-CC mechanism on the efficiency frontier, together with the reduction to the reassignment environment. The invariance property ensures that the auxiliary unacceptable school used in the reduction leaves the mechanism's outcome on the original problem unchanged.

DA-HC is implementable using exactly the information already collected by the Ministry: submitted ROLs, school capacities, the official geographic hierarchy, the official school ordering, and administrative priority rankings. It does not require changing the set of allowable reports or collecting new information from teachers. The reform is therefore targeted: it replaces the rejection rule used after regional applications while preserving the existing reporting language and other institutional details.

7 Empirical Relevance

We complement the theoretical results using administrative data from the Italian Ministry of Education on the universe of applications for the teacher reassignment procedure in the school years 2019–20, 2020–21, and 2021–22. The data include the geographical hierarchy, schools' capacities, submitted teacher ROLs, teacher scores, information about special-priority eligibility, and assignment outcomes.[27]

The empirical analysis has two objectives. First, we document that regional entries are a central feature of submitted ROLs, and hence that the hierarchical indifferences captured by the model are empirically relevant. Second, we evaluate DA-HC on submitted ROLs and compare its performance with the benchmark DA-STB mechanism and the current mechanism DA-HP.

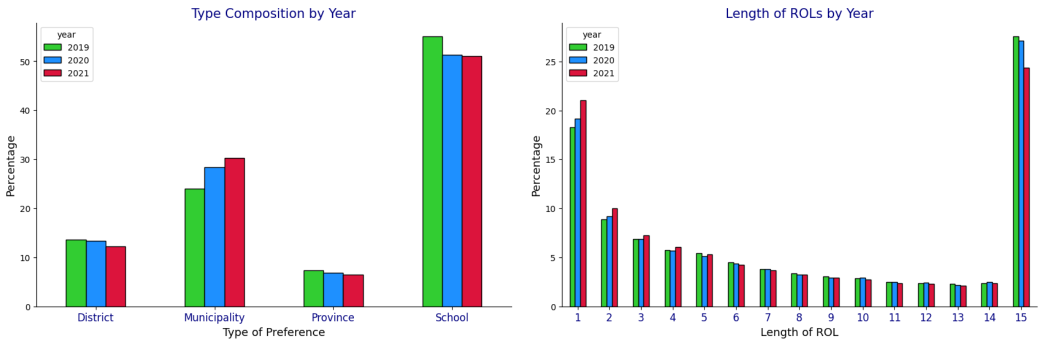
7.1 Regional Entries in Submitted ROLs

Regional entries are common. Teachers may submit an ROL with up to 15 items. Approximately 27 percent of applications include all 15 items, so 73 percent of applications do not use all available

[27]Appendix B.3 describes the data sources and construction in detail. As explained there, school priorities are based on teacher scores and other administrative criteria. We reconstruct priorities from the criteria observed in the data, but some criteria are not observed precisely. Therefore, the simulated mechanism outcomes, including priority violations under DA-HP, should be interpreted as based on reconstructed priorities rather than the Ministry's exact internal priority orderings.

slots. Around 20 percent of applications report only one item. On average, ROLs include 7–8 items. Across the three school years, 50–55 percent of ranked items are schools, 25–30 percent are municipalities, 10–12 percent are districts, and 6–7 percent are provinces (Figure 1). Thus, nearly half of all ranked items correspond to regions rather than individual schools.

Figure 1: Rank-Order Lists



(a) Type composition

(b) Length

Notes: Panel (a) shows the percentage of schools and regions (municipalities, districts, and provinces) among all items in submitted ROLs. Panel (b) shows the percentage of ROLs of each length.

Regional entries are not confined to the bottom of lists. Across years, regions account for 19–24 percent of first-ranked items, 27–34 percent of second-ranked items, and 32–38 percent of third-ranked items (Figure 2). Moreover, the frequency of regional entries is fairly stable across the three school years.

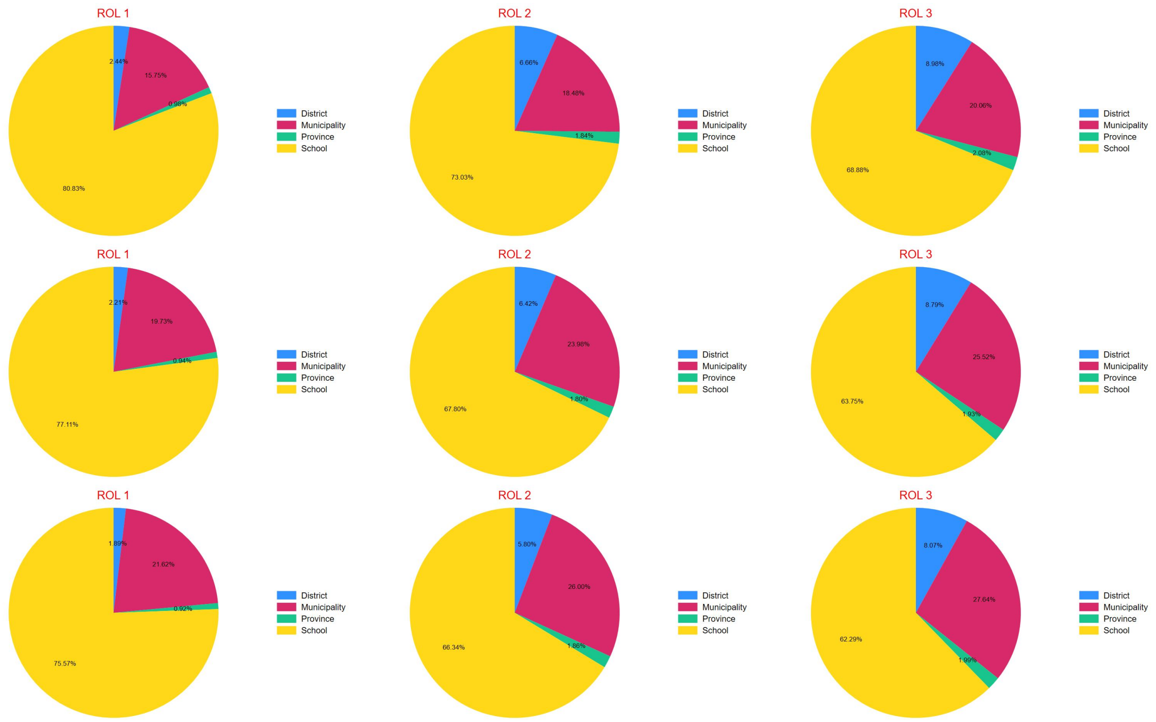
Figure 3 documents the geographic imbalance highlighted in the introduction. Remaining vacancies are concentrated mainly in northern provinces, while applications are concentrated more heavily in southern provinces. This pattern is consistent with regional entries capturing substantive geographic preferences.

These patterns show that regional entries are not merely residual options. Teachers frequently use the geographic reporting language to express indifferences among schools, including near the top of their ROLs. Thus, the scope for coordinated choice is empirically meaningful.

7.2 Assignment Improvements under DA-HC

We next compare DA-HC with DA-STB using submitted ROLs. For computational and data-construction reasons, we focus on the 2019 preschool-teacher subsample and geographic mobility

Figure 2: ROL Type Composition by Rank



Notes: The figure shows the distribution of single schools and regions among first-ranked items, second-ranked items, and third-ranked items in 2019–20, 2020–21, and 2021–22.

applications.[28] Preschool teachers constitute roughly 12–13 percent of all applications (Table 1).[29]

Table 1: Applications by School Type

School Type	*2019–20*	*2020–21*	*2021–22*
Preschool	12.48%	12.95%	12.76%
Primary School	27.77%	28.87%	29.34%
Middle School	18.90%	18.07%	17.13%
High School	40.85%	40.11%	40.78%
Total Applications	129,803	108,677	87,454

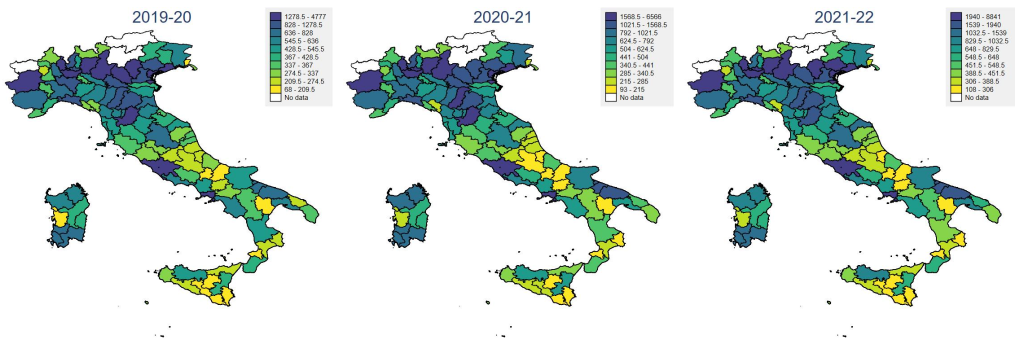
Notes: This table reports the fraction of applications by school type. We report data for school years 2019–20, 2020–21, and 2021–22.

In this subsample, 3.87 percent of teachers receive an assignment they prefer under DA-HC relative to DA-STB, while no teacher receives an assignment she prefers under DA-STB relative

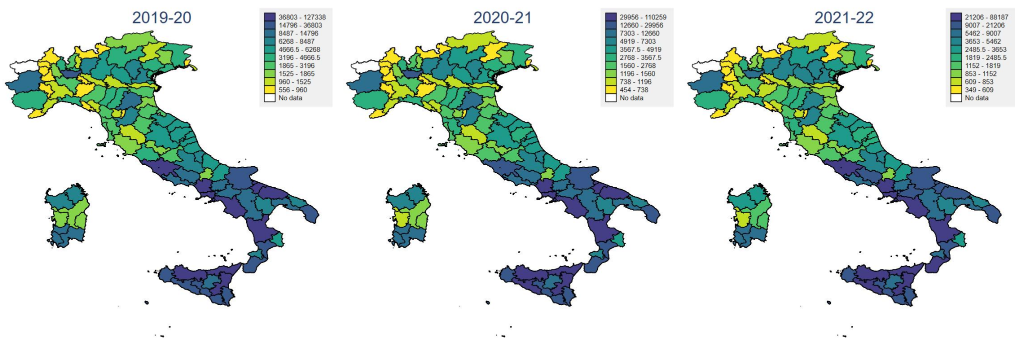
[28] Because of professional mobility, for example a teacher moving from Math to Economics, it is complicated to consider all school-field pairs for all types of schools. We therefore focus on geographic transfers and exclude professional mobility. See Section B.1 for different types of mobility. This simplification makes different school types and fields independent markets.

[29] Preschools have fewer types than primary schools, middle schools, and high schools, since the only distinction is between normal and special educational needs positions. Hence, focusing on preschool teachers is more convenient computationally and for data preparation.

Figure 3: Geographic Distribution of Vacancies and Applications



(a) Remaining vacancies



(b) Applications received

Notes: Panel (a) shows the distribution of remaining vacancies available for new teachers by province and school year. Panel (b) shows the distribution of applications received by province.

to DA-HC. This is consistent with Theorem 3. The gains arise because DA-HC can use regional entries more flexibly than DA-STB: teachers who are indifferent among schools in a region can be assigned to different schools within that region in a way that creates room for teachers with more refined preferences, without violating priorities.

This exercise is conservative in one respect. The submitted ROLs were generated under the current mechanism, which is manipulable. If teachers manipulate, they typically have incentives to replace broad regional entries with more specific schools or finer regions. Thus, submitted ROLs may understate the prevalence of weak preferences. Since regional indifferences are precisely the channel through which DA-HC improves upon DA-STB, the gains measured from submitted ROLs may understate the potential gains under truthful reporting.[30]

We also compare DA-HC with the current mechanism, DA-HP. In the 2019 preschool-teacher subsample, DA-HP generates priority violations for 1,310 teachers, corresponding to 12.9 percent of teachers in the subsample. Under DA-HC, this number is zero, as predicted by theory. On the other hand, there is no Pareto dominance relation between the two mechanisms.[31] Empirically, 37.7 percent of teachers prefer their DA-HP assignment to their DA-HC assignment, 0.4 percent prefer their DA-HC assignment to their DA-HP assignment, and the remaining teachers are indifferent. These findings point to an efficiency–fairness trade-off between DA-HP and DA-HC, despite the absence of a Pareto dominance relation.

Our simulation results in Appendix A.9 provide further evidence that DA-HC can eliminate a substantial number of priority violations generated by the current mechanism, DA-HP, while also delivering sizable welfare gains relative to the alternative priority-respecting benchmark, DA-STB. The share of students whose priorities are violated under DA-HP increases with the degree of preference correlation across students and can exceed 40 percent. By contrast, both DA-HC and DA-STB generate no priority violations. At the same time, the share of students who strictly prefer their DA-HC assignment to their DA-STB assignment decreases with the degree of preference correlation across students and can exceed 10 percent. At moderate levels of preference correlation, where both effects are simultaneously pronounced, DA-HC eliminates more than 10 percent of the priority violations generated by DA-HP while improving the assignments of more than 4 percent of students relative to DA-STB.

[30] Estimating teachers' latent preferences is beyond the scope of this paper. Because the current mechanism is neither strategy-proof nor stable, such an exercise would require modelling application behavior under complex dynamic incentives, incomplete information about vacancies, and a nontrivial portfolio-choice problem. Existing preference-estimation approaches for school choice and college admissions, such as Agarwal and Somaini (2018), Agarwal and Somaini (2020), Calsamiglia et al. (2020), and Fack et al. (2019), cannot be directly applied in our setting.

[31] For instance, in Example 2, teacher t_1 prefers her DA-HC assignment to her DA-HP assignment, whereas teacher t_2 prefers her DA-HP assignment to her DA-HC assignment.

8 Conclusion

This paper studies priority-based assignment when agents may have weak preferences. We introduce *Deferred Acceptance with Coordinated Choice* (DA-CC), a class of mechanisms that use indifferences to improve assignments while preserving fairness and strategy-proofness. DA-CC mechanisms lie on the efficiency frontier of fair and strategy-proof mechanisms. By contrast, the standard benchmark, *Deferred Acceptance with Simple Tie-Breaking* (DA-STB), need not.

We apply the framework to centralized teacher reassignment in Italy. In this system, teachers can rank geographic regions, thereby expressing indifference among schools. We show that the current mechanism can violate priority rights and create incentives for strategic reporting. We propose *Deferred Acceptance with Hierarchical Choice* (DA-HC), which preserves the existing reporting language, eliminates these priority violations, and Pareto improves upon DA-STB.

The broader implication is that weak preferences are not merely a complication to be resolved by tie-breaking. They can be a source of useful flexibility. Mechanisms that process this flexibility directly can improve assignments without compromising fairness or strategy-proofness.

References

Abdulkadiroğlu, Atila, Parag A Pathak, and Alvin E Roth (2009). "Strategy-proofness versus efficiency in matching with indifferences: Redesigning the NYC high school match". *American Economic Review* 99.5, pp. 1954–1978.

Abdulkadiroğlu, Atila and Tayfun Sönmez (2003). "School choice: A mechanism design approach". *American Economic Review* 93.3, pp. 729–747.

Afacan, Mustafa Oğuz and Umut Dur (2023). "Strategy-proof size improvement: is it possible?" *The Scandinavian Journal of Economics* 125.2, pp. 321–338.

Agarwal, Nikhil and Paulo Somaini (2018). "Demand analysis using strategic reports: An application to a school choice mechanism". *Econometrica* 86.2, pp. 391–444.

— (2020). "Revealed preference analysis of school choice models". *Annual Review of Economics* 12.1, pp. 471–501.

Alcalde-Unzu, Jorge and Elena Molis (2011). "Exchange of indivisible goods and indifferences: The top trading absorbing sets mechanisms". *Games and Economic Behavior* 73.1, pp. 1–16.

Alva, Samson and Vikram Manjunath (2020). "The impossibility of strategy-proof, Pareto efficient, and individually rational rules for fractional matching". *Games and Economic Behavior* 119, pp. 15–29.

Araujo, M Caridad, Pedro Carneiro, Yyannú Cruz-Aguayo, and Norbert Schady (2016). "Teacher quality and learning outcomes in kindergarten". *The Quarterly Journal of Economics* 131.3, pp. 1415–1453.

Aygün, Orhan and Tayfun Sönmez (2013). "Matching with contracts: Comment". *American Economic Review* 103.5, pp. 2050–2051.

Aziz, Haris, Md. Shahidul Islam, and Szilvia Pápai (2025). "Strategyproof Maximum Matching under Dichotomous Agent Preferences". *arXiv:2502.09962*.

Balinski, Michel and Tayfun Sönmez (1999). "A tale of two mechanisms: student placement". *Journal of Economic Theory* 84.1, pp. 73–94.

Bates, Michael, Michael Dinerstein, Andrew C Johnston, and Isaac Sorkin (2025). "Teacher labor market policy and the theory of the second best". *The Quarterly Journal of Economics* 140.2, pp. 1417–1469.

Berge, Claude (1957). "Two theorems in graph theory". *Proceedings of the National Academy of Sciences* 43.9, pp. 842–844.

Biasi, Barbara, Chao Fu, and John Stromme (2021). "Equilibrium in the market for public school teachers: District wage strategies and teacher comparative advantage". *NBER Working Paper* 28530.

Bobba, Matteo, Tim Ederer, Gianmarco Leon-Ciliotta, Christopher Neilson, and Marco G Nieddu (2021). "Teacher compensation and structural inequality: Evidence from centralized teacher school choice in Perú". *NBER Working Paper* 29068.

Calsamiglia, Caterina, Chao Fu, and Maia Güell (2020). "Structural estimation of a model of school choices: The boston mechanism versus its alternatives". *Journal of Political Economy* 128.2, pp. 642–680.

Cechlárová, Katarína, Tamás Fleiner, David F. Manlove, and Iain McBride (2016). "Stable matchings of teachers to schools". *Theoretical Computer Science* 653, pp. 15–25.

Cechlárová, Katarína, Tamás Fleiner, David F Manlove, Iain Mcbride, and Eva Potpinková (2015). "Modelling practical placement of trainee teachers to schools". *Central European Journal of Operations Research* 23, pp. 547–562.

Chetty, Raj, John N Friedman, and Jonah E Rockoff (2014). "Measuring the impacts of teachers II: Teacher value-added and student outcomes in adulthood". *American economic review* 104.9, pp. 2633–2679.

Combe, Julien, Umut Dur, Olivier Tercieux, Camille Terrier, and M Utku Ünver (2025). "Market design for distributional objectives in (re) assignment: An application to improve the distribution of teachers in schools". *Working Paper*.

Combe, Julien, Olivier Tercieux, and Camille Terrier (2022). "The design of teacher assignment: Theory and evidence". *The Review of Economic Studies* 89.6, pp. 3154–3222.

Davis, Jonathan MV (forthcoming). "Labor Market Design Can Improve Match Outcomes: Evidence from Matching Teach For America Teachers to Schools." *Journal of Political Economy Microeconomics*.

De Haan, Monique, Pieter A Gautier, Hessel Oosterbeek, and Bas Van der Klaauw (2023). "The performance of school assignment mechanisms in practice". *Journal of Political Economy* 131.2, pp. 388–455.

Doğan, Battal and Madhav Raghavan (2024). "Equitable Allocation of Vaccines in a Supply Network: Application to COVAX". *Working Paper*.

Dur, Umut and Onur Kesten (2019). "Sequential versus simultaneous assignment systems and two applications". *Economic Theory* 68.2, pp. 251–283.

Ederer, Tim (2023). "Labor market dynamics and teacher spatial sorting". *Working Paper*.

Elacqua, Gregory, Leidy Gómez, Thomas Krussig, Luana Marotta, Carolina Méndez, and Christopher A Neilson (2022). *The potential of smart matching platforms in teacher assignment: The case of ecuador*. Tech. rep. IDB Working Paper Series.

Elacqua, Gregory and Leonardo Rosa (2025). "Teacher transfers and the disruption of teacher staffing in the City of Sao Paulo". *International Journal of Educational Development* 117, p. 103353.

Elacqua, Gregory, Anne Sofie Westh Olsen, and Santiago Velez-Ferro (2021). *The market design approach to teacher assignment: Evidence from ecuador*. Tech. rep. IDB Working Paper Series.

Erdil, Aytek (2014). "Strategy-proof stochastic assignment". *Journal of Economic Theory* 151, pp. 146–162.

Erdil, Aytek and Haluk Ergin (2008). "What's the matter with tie-breaking? Improving efficiency in school choice". *American Economic Review* 98.3, pp. 669–689.

Erdil, Aytek and Haluk Ergin (2017). “Two-sided matching with indifferences”. *Journal of Economic Theory* 171, pp. 268–292.
Fack, Gabrielle, Julien Grenet, and Yinghua He (2019). “Beyond truth-telling: Preference estimation with centralized school choice and college admissions”. *American Economic Review* 109.4, pp. 1486–1529.
Gale, David and Lloyd S Shapley (1962). “College admissions and the stability of marriage”. *The American Mathematical Monthly* 69.1, pp. 9–15.
Hatfield, John William and Scott Duke Kominers (2019). “Hidden Substitutes”. *Working Paper*.
Hatfield, John William and Paul R Milgrom (2005). “Matching with contracts”. *American Economic Review* 95.4, pp. 913–935.
Jaramillo, Paula and Vikram Manjunath (2012). “The difference indifference makes in strategy-proof allocation of objects”. *Journal of Economic Theory* 147.5, pp. 1913–1946.
Klein, Thilo and Philip vom Baur (2019). “Matching practices for trainee teachers - Germany”. *Matching in Practice-28*.
Laverde, Mariana, Elton Mykerezi, Aaron Sojourner, and Aradhya Sood (2026). “Match Effects and the Gains from Alternative Job Assignments: Evidence from a Teacher Labor Market”. *IZA Discussion Paper, No. 18397*.
Li, Mengling (2020). “Ties matter: Improving efficiency in course allocation by allowing ties”. *Journal of Economic Behavior & Organization* 178, pp. 354–384.
Luschei, Thomas F and Martin Carnoy (2010). “Educational production and the distribution of teachers in Uruguay”. *International Journal of Educational Development* 30.2, pp. 169–181.
Manlove, David F, Robert W Irving, Kazuo Iwama, Shuichi Miyazaki, and Yasufumi Morita (2002). “Hard variants of stable marriage”. *Theoretical Computer Science* 276.1-2, pp. 261–279.
Mishra, Debasis, Soumendu Sarkar, Arunava Sen, Jay Sethuraman, and Sonal Yadav (2025). “Teacher transfers: equalizing deficits across schools”. *arXiv:2510.18708*.
Pathak, Parag A. and Tayfun Sönmez (Sept. 2008). “Leveling the Playing Field: Sincere and Sophisticated Players in the Boston Mechanism”. *American Economic Review* 98.4, pp. 1636–1652.
Pereyra, Juan Sebastián (2013). “A dynamic school choice model”. *Games and Economic Behavior* 80, pp. 100–114.
Rockoff, Jonah E (2004). “The impact of individual teachers on student achievement: Evidence from panel data”. *American economic review* 94.2, pp. 247–252.
Roth, Alvin E. (1991). “A Natural Experiment in the Organization of Entry-Level Labor Markets: Regional Markets for New Physicians and Surgeons in the United Kingdom”. *American Economic Review* 81.3, pp. 415–440.
— (2002). “The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics”. *Econometrica* 70.4, pp. 1341–1378.
— (2008). “What Have We Learned from Market Design?” *Economic Journal* 118.527, pp. 285–310.
Saban, Daniela and Jay Sethuraman (2013). “House allocation with indifferences: a generalization and a unified view”. *Proceedings of the fourteenth ACM conference on Electronic Commerce*, pp. 803–820.
Silhol, Julien and Lionel Wilner (2022). “Teachers’ Desired Mobility to Disadvantaged Schools: Do Financial Incentives Matter?” *CESifo Working Papers* N. 9906.
Sönmez, Tayfun (2023). “Minimalist Market Design: A Framework for Economists with Policy Aspirations”. *arXiv:2401.00307*.
Terrier, Camille (2014). “Matching Practices for secondary public school teachers–France”. *Matching in Practice*.
Tomás, Ana Paula (2017). “House allocation problems with existing tenants and priorities for teacher recruitment”. *International Conference on Current Trends in Theory and Practice of Informatics*. Springer, pp. 479–492.

Appendix A Proofs

A.1 Proof of Proposition 1

Consider the following problem. Let $T = \{t_1, t_2, t_3\}$, $S = \{s_1, s_2\}$, and $q_{s_1} = q_{s_2} = 1$. Preferences and priorities are as follows; schools omitted from a teacher's column are ranked below the outside option $\emptyset$.

R_{t_1}	R_{t_2}	R_{t_3}	$\succ_{s_1}$	$\succ_{s_2}$
$\{s_1, s_2\}$	s_1	s_2	t_1	t_1
$\emptyset$	$\emptyset$	$\emptyset$	t_3	t_2
			t_2	t_3

Suppose that φ is constrained Pareto efficient. In particular, φ is fair. Since t_1 strictly prefers both s_1 and s_2 to $\emptyset$ and has top priority at both schools, fairness implies $\varphi_{t_1}(R) \neq \emptyset$. Hence either $\varphi_{t_1}(R) = s_1$ or $\varphi_{t_1}(R) = s_2$.

Suppose first that $\varphi_{t_1}(R) = s_1$. Then fairness and constrained Pareto efficiency imply $\varphi_{t_2}(R) = \emptyset$ and $\varphi_{t_3}(R) = s_2$. Now consider the report R'_{t_2} under which $s_1 \; P'_{t_2} \; s_2 \; P'_{t_2} \; \emptyset$. Let $R' = (R'_{t_2}, R_{-t_2})$. At R', the unique constrained Pareto efficient matching assigns t_1 to s_2, t_2 to s_1, and t_3 to $\emptyset$. Indeed, the fair matching assigning t_1 to s_1, t_2 to s_2, and t_3 to $\emptyset$ is Pareto dominated by this matching, since t_1 is indifferent between s_1 and s_2 while t_2 strictly prefers s_1 to s_2. Therefore, $\varphi_{t_2}(R') = s_1$. Thus, when the true profile is R, teacher t_2 can profitably misreport R'_{t_2} and obtain s_1 instead of $\emptyset$, contradicting strategy-proofness. The case $\varphi_{t_1}(R) = s_2$ follows by a symmetric argument.

A.2 Proof of Proposition 2

Consider the following problem. Let $T = \{t_1, t_2, t_3\}$, $S = \{s_1, s_2, s_3\}$, and $q_{s_1} = q_{s_2} = q_{s_3} = 1$. Preferences and priorities are as follows; schools omitted from a teacher's column are ranked below the outside option $\emptyset$.

R_{t_1}	R_{t_2}	R_{t_3}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$\{s_1, s_2\}$	s_1	s_2	t_1	t_1	t_1
$\emptyset$	$\emptyset$	$\emptyset$	t_3	t_2	t_2
			t_2	t_3	t_3

Suppose that φ is constrained size efficient. In particular, φ is fair. Since t_1 strictly prefers both s_1 and s_2 to $\emptyset$ and has top priority at both schools, fairness implies $\varphi_{t_1}(R) \neq \emptyset$. Hence either

$\varphi_{t_1}(R) = s_1$ or $\varphi_{t_1}(R) = s_2$.

Suppose first that $\varphi_{t_1}(R) = s_1$. Then fairness and constrained size efficiency imply $\varphi_{t_2}(R) = \emptyset$ and $\varphi_{t_3}(R) = s_2$. Now consider another profile R' such that $R'_{t_1} = R_{t_1}$, $R'_{t_2} = R_{t_2}$, and $s_2 \; P'_{t_3} \; s_3 \; P'_{t_3} \; \emptyset$. At R', the unique maximum-size fair matching assigns t_1 to s_2, t_2 to s_1, and t_3 to s_3. Thus constrained size efficiency requires $\varphi_{t_1}(R') = s_2$, $\varphi_{t_2}(R') = s_1$, and $\varphi_{t_3}(R') = s_3$.

But then, when the true profile is R', teacher t_3 can report R_{t_3}. The resulting reported profile is R, and by the maintained case $\varphi_{t_3}(R) = s_2$. Since $s_2 \; P'_{t_3} \; s_3$, this is a profitable manipulation, contradicting strategy-proofness. The case $\varphi_{t_1}(R) = s_2$ follows by a symmetric argument.

A.3 Proof of Proposition 3

Towards a contradiction, suppose that there exists a fair and strategy-proof mechanism φ' that Pareto improves over φ. Then there exists a problem R such that $\varphi'(R)$ Pareto dominates $\varphi(R)$. Hence, for some teacher $t \in T$, $\varphi'_t(R) \; P_t \; \varphi_t(R)$. Let $\varphi_t(R) = x$ and $\varphi'_t(R) = s'$. Thus $s' \; P_t \; x$.

Since φ is constrained Pareto-size efficient, $\varphi(R)$ is fair; in particular, teacher t is assigned either an acceptable school or the outside option. Hence $x \; R_t \; \emptyset$. By transitivity, $s' \; P_t \; \emptyset$, so s' is an acceptable school for t.

Let R'_t be a weak preference relation such that s' is the unique acceptable school:

$$s' \; P'_t \; \emptyset \; P'_t \; y \quad \text{for every } y \in S \setminus \{s'\}.$$

Let $R' = (R'_t, R_{-t})$.

We first show that $\varphi_t(R') \neq s'$. Suppose instead that $\varphi_t(R') = s'$. Then, at problem R, teacher t could report R'_t and obtain s' instead of x. Since $s' \; P_t \; x$, this would be a profitable manipulation, contradicting strategy-proofness of φ. Therefore, $\varphi_t(R') \neq s'$. Since s' is the only acceptable school under R'_t, teacher t is not assigned to an acceptable school under $\varphi(R')$.

We next show that $\varphi'_t(R') = s'$. Suppose instead that $\varphi'_t(R') \neq s'$. Then, at problem R', teacher t could report R_t. The resulting reported profile would be R, so teacher t would obtain $\varphi'_t(R) = s'$. Since s' is strictly preferred under R'_t to every other possible assignment, this would be a profitable manipulation, contradicting strategy-proofness of φ'. Therefore, $\varphi'_t(R') = s'$. Hence teacher t is assigned to an acceptable school under $\varphi'(R')$.

Because φ' Pareto improves over φ, $\varphi'(R')$ Pareto dominates $\varphi(R')$. Moreover, teacher t is assigned to an acceptable school under $\varphi'(R')$ but not under $\varphi(R')$.

For every teacher $j \neq t$, preferences are unchanged from R to R'. Since $\varphi'(R')$ Pareto dominates $\varphi(R')$, any such teacher who is assigned to an acceptable school under $\varphi(R')$ is also assigned to an acceptable school under $\varphi'(R')$. Therefore, $\varphi'(R')$ assigns strictly more teachers to acceptable schools than $\varphi(R')$ does. Hence $\varphi'(R')$ size dominates $\varphi(R')$.

Since φ' is fair, $\varphi'(R')$ is a fair matching. Thus, at problem R', there exists a fair matching, namely $\varphi'(R')$, that both Pareto dominates and size dominates $\varphi(R')$. This contradicts constrained Pareto-size efficiency of φ. Therefore, no fair and strategy-proof mechanism Pareto improves over φ.

A.4 Proof of Proposition 4

Fix an ordering $\triangleright$ on S and an application profile A. Let $x_1 \vdash \cdots \vdash x_m$ be the $\vdash$-ordering of admissible pairs and let $\kappa = \sum_{s \in S} q_s$ denote total capacity. We assign weights recursively from the bottom of the ordering. First set $w(x_{m-1}) - w(x_m) = 1$. Then, for each $k = 2, \ldots, m-1$, set

$$w(x_{m-k}) - w(x_{m-k+1}) = \kappa\big[w(x_{m-k+1}) - w(x_m)\big].$$

Note that $w(x_1) - w(x_m) = \sum_{k=1}^{m-1}\big[w(x_k) - w(x_{k+1})\big]$ is fully determined by the above differences. After these differences are defined, choose the additive constant so that

$$w(x_m) = \kappa\big[w(x_1) - w(x_m)\big].$$

These weights implement the lexicographic order induced by $\vdash$: a larger-size matching has greater total weight than any smaller-size matching, and among matchings of the same size, the first admissible pair according to $\vdash$ on which they differ determines which matching has greater total weight. Hence $C^{\triangleright}(A)$ coincides with the maximum-weight matching in the weighted bipartite graph induced by these weights. We use this equivalence to prove each property in turn.

Maximum-size. Suppose towards a contradiction that $C^{\triangleright}(A)$ is not maximum-size. Then there exists a matching μ' respecting A with strictly more matched teachers. Since each admissible pair receives weight at least $w(x_m) = \kappa \cdot [w(x_1) - w(x_m)]$, and the maximum weight difference between any two pairs is $w(x_1) - w(x_m)$ while $\kappa - 1$ is an upper bound on the number of pairs any matching can contain, μ' has strictly greater total weight than $C^{\triangleright}(A)$, contradicting that $C^{\triangleright}(A)$ is the maximum-weight matching.

Priority-respecting. Suppose towards a contradiction that $C^{\triangleright}(A)$ is not priority-respecting. Then there exist $t, t' \in T$ and $s \in A(t)$ such that $C^{\triangleright}(A)(t) = \emptyset$, $t' \in C^{\triangleright}(A)(s)$, and $t \succ_s t'$. By the $\vdash$-ordering, $(t, s) \vdash (t', s)$, so $w(t, s) > w(t', s)$. The matching obtained from $C^{\triangleright}(A)$ by assigning s to t and leaving t' unassigned has strictly greater weight, contradicting that $C^{\triangleright}(A)$ is the maximum-weight matching.

Substitutability. The choice rule $C^{\rhd}$ admits a maximum-weight matching formulation. One can translate this formulation into an endowed assignment valuation and apply results from Hatfield and Milgrom (2005), which establishes substitutes for endowed assignment valuations. We give a direct, elementary proof instead.

Fix an application profile A and a teacher t_0, and let $A^- = A_{-t_0}$ be obtained from A by removing t_0's application. Let $\mu = C^{\rhd}(A)$ and $\mu^- = C^{\rhd}(A^-)$. We show that, for each teacher $t_1 \neq t_0$, if $\mu(t_1) \neq \emptyset$, then $\mu^-(t_1) \neq \emptyset$.

Suppose, towards a contradiction, that there exists $t_1 \neq t_0$ such that $\mu(t_1) = s_1$ and $\mu^-(t_1) = \emptyset$. Since $s_1 \in A(t_1) = A^-(t_1)$ and μ^- is maximum-size, school s_1 must be full under μ^-. Moreover, since μ^- is priority-respecting, every teacher assigned to s_1 under μ^- has higher priority than t_1 at s_1.

Since t_1 is assigned to s_1 under μ but not under μ^-, there exists a teacher t_2 such that $\mu^-(t_2) = s_1$ and $\mu(t_2) \neq s_1$. By above, $t_2 \succ_{s_1} t_1$. Teacher t_2 must be assigned under μ. Otherwise, replacing (t_1, s_1) in μ with (t_2, s_1) would preserve size and produce a lexicographically better maximum-size matching than μ, contradicting the definition of $C^{\rhd}(A)$. Let $\mu(t_2) = s_2$. Then $s_2 \neq s_1$.

If s_2 had an empty seat under μ^-, then starting from μ^-, assigning t_1 to s_1 and moving t_2 from s_1 to s_2 would increase the size of μ^-, contradicting maximum-size. Hence s_2 is full under μ^-. Since t_2 is assigned to s_2 under μ but not under μ^-, there exists a teacher t_3 such that $\mu^-(t_3) = s_2$ and $\mu(t_3) \neq s_2$.

Continue similarly. Suppose teachers $t_1, \ldots, t_{i+1}$ and schools $s_1, \ldots, s_i$ have been identified so that, for each $j \in \{1, \ldots, i\}$, $\mu(t_j) = s_j$ and $\mu^-(t_{j+1}) = s_j$, with $\mu(t_{j+1}) \neq s_j$. If $\mu(t_{i+1}) = \emptyset$, stop. Otherwise, let $\mu(t_{i+1}) = s_{i+1}$. If s_{i+1} had an empty seat under μ^-, then starting from μ^- and assigning t_j to s_j for each $j \in \{1, \ldots, i+1\}$ would increase the size of μ^-: the teachers $t_2, \ldots, t_{i+1}$ are moved from $s_1, \ldots, s_i$ to $s_2, \ldots, s_{i+1}$, and t_1 is newly assigned to s_1. This contradicts maximum-size of μ^-. Hence s_{i+1} is full under μ^-. Since t_{i+1} is assigned to s_{i+1} under μ but not under μ^-, there exists a teacher t_{i+2} such that $\mu^-(t_{i+2}) = s_{i+1}$ and $\mu(t_{i+2}) \neq s_{i+1}$.

Since the set of teachers is finite, this construction either reaches a teacher t_{n+1} such that $\mu(t_{n+1}) = \emptyset$, or it reaches the first repeated teacher, which creates a cycle.

First suppose that the construction reaches a teacher t_{n+1} such that $\mu(t_{n+1}) = \emptyset$. Then, for each $i \in \{1, \ldots, n\}$, we have $\mu(t_i) = s_i$ and $\mu^-(t_{i+1}) = s_i$, while $\mu^-(t_1) = \emptyset$.

Let $P = \{(t_i, s_i) : i = 1, \ldots, n\}$ and $P^- = \{(t_{i+1}, s_i) : i = 1, \ldots, n\}$. Construct $\widehat{\mu}^-$ from μ^- by replacing the pairs in P^- with the pairs in P, leaving all other assignments unchanged. This matching respects A^- and has the same size as μ^-. Construct $\widehat{\mu}$ from μ by replacing the pairs in P with the pairs in P^-, leaving all other assignments unchanged. This matching respects A and has the same size as μ.

Let x be the highest $\vdash$-order pair in the symmetric difference $(P \setminus P^-) \cup (P^- \setminus P)$. If $x \in P$, then $\widehat{\mu}^-$ is lexicographically preferred to μ^- among maximum-size matchings respecting A^-, contradicting the definition of $C^{\triangleright}(A^-)$. If $x \in P^-$, then $\widehat{\mu}$ is lexicographically preferred to μ among maximum-size matchings respecting A, contradicting the definition of $C^{\triangleright}(A)$.

It remains to consider the case in which the construction creates a cycle. Then there exist indices $a \leq b$ such that $\mu(t_i) = s_i$ for each $i \in \{a, \ldots, b\}$, $\mu^-(t_{i+1}) = s_i$ for each $i \in \{a, \ldots, b-1\}$, and $\mu^-(t_a) = s_b$. Let $P = \{(t_i, s_i) : i = a, \ldots, b\}$ and $P^- = \{(t_{i+1}, s_i) : i = a, \ldots, b-1\} \cup \{(t_a, s_b)\}$. Construct $\widehat{\mu}^-$ from μ^- by replacing the pairs in P^- with the pairs in P, leaving all other assignments unchanged. This matching respects A^- and has the same size as μ^-. Construct $\widehat{\mu}$ from μ by replacing the pairs in P with the pairs in P^-, leaving all other assignments unchanged. This matching respects A and has the same size as μ.

Let x be the highest $\vdash$-order pair in the symmetric difference $(P \setminus P^-) \cup (P^- \setminus P)$. If $x \in P$, then $\widehat{\mu}^-$ is lexicographically preferred to μ^- among maximum-size matchings respecting A^-, contradicting the definition of $C^{\triangleright}(A^-)$. If $x \in P^-$, then $\widehat{\mu}$ is lexicographically preferred to μ among maximum-size matchings respecting A, contradicting the definition of $C^{\triangleright}(A)$. Hence $C^{\triangleright}$ is substitutable.

A.5 Proof of Theorem 1

Let C^* be an arbitrary choice rule satisfying the three properties. In what follows, by the DA-CC mechanism, we refer to the DA-CC mechanism induced by C^*.

We associate each problem with a matching with contracts problem (Hatfield and Milgrom, 2005) as follows. The set of doctors coincides with the set of teachers. There is a single hospital, and all schools can be thought of as different divisions of the only hospital. The set of contractual terms is the collection of all non-empty subsets of schools, $2^S \setminus \{\emptyset\}$. Each contract (t, S') specifies a teacher $t \in T$ and a non-empty set of schools $S' \in 2^S \setminus \{\emptyset\}$. Let $\mathcal{X} = T \times (2^S \setminus \{\emptyset\})$ denote the set of all possible contracts.

A set of contracts is **feasible** if each teacher appears in at most one contract.

A **choice function** Ch of the only hospital associates each $X \subseteq \mathcal{X}$ with a feasible set of chosen contracts $Ch(X) \subseteq X$.

A **partial choice function** associates each feasible set of contracts $X \subseteq \mathcal{X}$ with a feasible set of chosen contracts $Ch(X) \subseteq X$.

A **pseudo choice function** associates each $X \subseteq \mathcal{X}$ with a set of chosen contracts $Ch(X) \subseteq X$, where a teacher may appear in several chosen contracts, in contrast to a choice function.[32]

The choice rule C^* induces the following partial choice function Ch^{par}. For any feasible set of contracts $X \subseteq \mathcal{X}$, define $Ch^{par}(X)$ as follows. Consider the application profile A where, for each $t \in T$, $A(t) = \emptyset$ if there is no contract involving t in X, and $A(t) = S'$ if $(t, S') \in X$ for some $S' \in 2^S \setminus \{\emptyset\}$. Let

$$Ch^{par}(X) \equiv \{(t, S') \in X : C_t^*(A) \neq \emptyset\}.$$

Note that Ch^{par} satisfies the following properties on the domain of feasible sets of contracts.

Substitutes: If a contract is chosen from a feasible set of contracts, then it is still chosen if another contract is removed from the choice problem. This directly follows from the fact that C^* is substitutable.

Law of aggregate demand (LAD): If a contract is removed from a feasible set of contracts, then the number of chosen contracts does not increase. This directly follows from the fact that C^* is maximum-size.

Irrelevance of rejected contracts (IRC): If a rejected contract is removed from a feasible set of contracts, then the set of chosen contracts remains the same. This is because substitutes together with LAD imply IRC (Aygün and Sönmez, 2013).

We now construct a pseudo choice function Ch^{pse}. Roughly, this pseudo choice function operates the same as Ch^{par}, but treats different contracts of the same teacher as if they involved different teachers, and therefore may choose several contracts involving the same teacher. We formalize this below.

Associate each contract $(t, S') \in \mathcal{X}$ with an auxiliary teacher $t^*_{(t,S')}$, with distinct auxiliary teachers for distinct contracts. Let T^{aux} be the set of all auxiliary teachers. Note that $|T^{aux}| = |T| \times |2^S \setminus \{\emptyset\}|$. For each $s \in S$, define an auxiliary priority ordering $\succ_s^{aux}$ over T^{aux} that is consistent with $\succ_s$: for any $t^*_{(t,S')}, k^*_{(k,S'')} \in T^{aux}$, if $t \succ_s k$, then

$$t^*_{(t,S')} \succ_s^{aux} k^*_{(k,S'')}.$$

Take any $X \subseteq \mathcal{X}$. Let X^* be obtained from X by replacing each $(t, S') \in X$ with $(t^*_{(t,S')}, S')$. Observe that the choice procedure underlying Ch^{par} is well-defined for X^* as well, using the

[32] This concept was introduced by Hatfield and Kominers (2019). Note, however, that unlike in Hatfield and Kominers (2019), the original choice rule in our context is not defined on the domain of all possible sets of contracts, but only on the domain of feasible sets of contracts, which is similar to the approach in **dogan_erdil_2025**. Nevertheless, the results from Hatfield and Kominers (2019) still apply. For other recent studies using results from Hatfield and Kominers (2019), see, among others, **Shorrer_Psychology** and **aygun2017large**; **aygun_turhan_2020**.

auxiliary priorities. Define Ch^{pse} as follows: for each $(t, S') \in X$,

$$(t, S') \in Ch^{pse}(X) \quad \text{if and only if} \quad (t^*_{(t,S')}, S') \in Ch^{par}(X^*).$$

Equivalently, $(t, S') \in Ch^{pse}(X)$ if and only if $C^*_{t^*_{(t,S')}}(A^*) \neq \emptyset$, where A^* is the application profile such that, for each $t^*_{(t,S')} \in T^{aux}$, $A^*(t^*_{(t,S')}) = \emptyset$ if $(t^*_{(t,S')}, S') \notin X^*$, and $A^*(t^*_{(t,S')}) = S'$ otherwise.

Observe that Ch^{pse} is a substitutable completion (Hatfield and Kominers, 2019) of Ch^{par} that satisfies LAD.

We next define the cumulative offers (CO) algorithm.

Cumulative Offers (CO) Algorithm

Step 1: Each teacher proposes her top-ranked acceptable contract. If there is no proposal, then stop and return the resulting empty matching. Otherwise, let the hospital hold the contracts that Ch^{pse} chooses from those that have been proposed, and go to Step 2.

Step $k \geq 2$: Each teacher who is not included in a currently held contract proposes her next-best acceptable contract. If there is no proposal, then stop and return the pseudo matching consisting of the contracts held at the end of the previous step. Otherwise, let the hospital hold the contracts that Ch^{pse} chooses from the cumulative set of all proposals received since the beginning of Step 1, and go to Step $k + 1$.

The algorithm must eventually stop because no teacher proposes the same contract twice.

The Cumulative Offers (CO) mechanism returns, for each problem, the pseudo matching produced by the CO algorithm. The lemma below establishes the equivalence of DA-CC and CO, and in particular verifies that the outcome of CO is actually a matching.

Lemma 1. *The tentative assignments of the DA-CC mechanism and the CO mechanism coincide at every step. That is, for every $k \geq 1$, a teacher $t \in T$ applies to a set of schools $A^k(t)$ and is tentatively assigned to a school $s \in A^k(t)$ at the end of Step k in the DA-CC algorithm if and only if the contract $(t, A^k(t))$ is tentatively accepted in Step k of the CO algorithm.*

Proof. Given an arbitrary problem, consider the first steps of the DA-CC and CO algorithms. Teacher t applies to $A^1(t)$ in Step 1 of the DA-CC algorithm, and at the same time proposes $(t, A^1(t))$ in Step 1 of the CO algorithm. Since no teacher applies to multiple sets of schools at this step, and since C^* coincides with the associated pseudo choice function whenever no teacher applies to more than one set of schools, the tentative assignments of the DA-CC and CO mechanisms coincide at Step 1.

Suppose the statement holds up to some Step $k-1$. Consider Step k of the DA-CC and CO algorithms. Teacher t applies to $A^k(t)$ in Step k of the DA-CC algorithm, and at the same time proposes $(t, A^k(t))$ in Step k of the CO algorithm. If t has proposed multiple contracts since the first step of the CO algorithm, all contracts different from $(t, A^k(t))$ must have been rejected by Ch^{pse} in previous steps. Throughout the CO algorithm, the set of proposed contracts is weakly expanding. Therefore, IRC implies that the set of contracts chosen by Ch^{pse} from all contracts proposed since Step 1 is the same as the set chosen from the contracts proposed since Step 1 and not yet rejected.

The applicants from Step k, together with the tentatively accepted applicants in the DA-CC algorithm, coincide with the set of contracts that have been proposed and never rejected since Step 1 in the CO algorithm. Moreover, in Step k, no teacher is included in two different applications among the contracts that have been proposed and not yet rejected. Since C^* coincides with the associated pseudo choice function whenever no teacher applies to multiple sets of schools, the tentative assignments of the DA-CC and CO mechanisms coincide at Step k. □

DA-CC is fair and strategy-proof

Strategy-proofness follows from Theorem 3 of Hatfield and Kominers (2019). To establish fairness, consider an arbitrary problem and let μ be the outcome of the DA-CC mechanism for this problem. Take any teacher $t \in T$ and school $s \in S$ such that $s \; P_t \; \mu(t)$. Let X be the cumulative set of contracts in the last step of the CO algorithm. Since t must have applied to an indifference class containing s in the course of the DA-CC algorithm, and since DA-CC and CO coincide at every step, there exists $(t, S') \in X$ with $s \in S'$. Since C^* is maximum-size, Ch^{pse} always chooses a maximum-size matching as well. Therefore all seats at s must be allocated. Moreover, since C^* is priority-respecting, all seats at s must be allocated to teachers who have higher priority than t. Hence, DA-CC is fair.

DA-CC attains the efficiency frontier

We next show that the DA-CC outcome μ is constrained Pareto-size efficient, and therefore attains the efficiency frontier by Proposition 3. Suppose, towards a contradiction, that μ is not constrained Pareto-size efficient. Then there exists another matching μ' that is fair, Pareto dominates μ, and assigns more teachers to acceptable schools.

Since μ' assigns more teachers to acceptable schools than μ, and since μ' Pareto dominates

μ, there exists a teacher t_0 such that $\mu(t_0) = \emptyset$ and $\mu'(t_0) = s_1$ for some $s_1 \; P_{t_0} \; \emptyset$. Consider the directed comparison between μ and μ' that starts from this reassignment of t_0 to s_1. If s_1 has an empty seat under μ, the path stops. Otherwise, some teacher t_1 assigned to s_1 under μ is assigned under μ' to a different school s_2 with $s_2 \; R_{t_1} \; s_1$, because μ' Pareto dominates μ. If s_2 has an empty seat under μ, the path stops. Otherwise, continue in the same way. Since μ' assigns strictly more teachers to acceptable schools than μ, the standard augmenting-path argument implies that this process has a path ending at a school with an empty seat under μ.

Thus, there exist teachers $(t_0, t_1, \ldots, t_{m-1})$ and schools $(s_1, \ldots, s_m)$ such that

i. $s_1 \; P_{t_0} \; \emptyset$,

ii. for each $k \in \{1, \ldots, m-1\}$, $\mu(t_k) = s_k$ and $\mu'(t_k) = s_{k+1} \; R_{t_k} \; s_k$ with $s_{k+1} \neq s_k$, and

iii. $|\mu(s_m)| < q_{s_m}$.

Consider any $k \in \{1, \ldots, m-1\}$. Let S'_k be the set of schools associated with the contract of t_k that is held in the last step of the CO algorithm, so that $s_k \in S'_k$. If $s_{k+1} \; I_{t_k} \; s_k$, then $s_{k+1} \in S'_k$. If instead $s_{k+1} \; P_{t_k} \; s_k$, then there is a contract $(t_k, S''_k) \in X$ with $s_{k+1} \in S''_k$, because t_k must have proposed to the indifference class containing s_{k+1} before proposing to the class containing s_k. Similarly, there is a contract $(t_0, S''_0) \in X$ with $s_1 \in S''_0$.

Let X^* be obtained from X by replacing each $(t, S') \in X$ with $(t^*_{(t,S')}, S')$, and let A^* be the corresponding auxiliary application profile. By construction, $(t, S') \in Ch^{pse}(X)$ if and only if $C^*_{t^*_{(t,S')}}(A^*) \neq \emptyset$.

Now construct an alternative matching for A^* as follows. Assign the auxiliary copy of t_0 associated with a contract containing s_1 to s_1. For each $k \in \{1, \ldots, m-1\}$, assign an auxiliary copy of t_k associated with a contract containing s_{k+1} to s_{k+1}, and remove the auxiliary copy of t_k that was assigned to s_k under $C^*(A^*)$. Leave all other assignments unchanged. This alternative matching respects the application profile A^* and the capacity constraints. It assigns one more teacher than $C^*(A^*)$, because t_0 is newly assigned and the path ends at a school s_m with an empty seat under μ. This contradicts the maximum-size property of C^*. Therefore, the DA-CC outcome is constrained Pareto-size efficient.

A.6 Proof of Theorem 2

Take any strict ordering $\rhd$ on S, and consider an arbitrary problem R. We show that, in the DA-CC algorithm induced by $C^{\rhd}$, no teacher is ever rejected when her current application contains her DA-STB assignment.

Suppose, towards a contradiction, that this is false. Let k be the earliest step of the DA-CC algorithm at which some teacher t_1 is rejected while applying to a set $A^k(t_1)$ containing her DA-STB assignment. Let $s_1 = DA\text{-}STB_{t_1}(R)$. Since t_1 is rejected at Step k while $s_1 \in A^k(t_1)$, priority-respectingness of $C^{\rhd}$ implies that all seats of s_1 are assigned at $C^{\rhd}(A^k)$ to teachers with higher priority than t_1 at s_1.

Since t_1 is assigned to s_1 under DA-STB but not under $C^{\rhd}(A^k)$, there exists a teacher t_2 such that $C^{\rhd}_{t_2}(A^k) = s_1$ and $DA\text{-}STB_{t_2}(R) \neq s_1$. Since t_2 is assigned to s_1 under $C^{\rhd}(A^k)$, school s_1 is acceptable for t_2. If $DA\text{-}STB_{t_2}(R) = \emptyset$, then fairness of DA-STB, together with the fact that t_1 is assigned to s_1 under DA-STB, would imply $t_1 \succ_{s_1} t_2$, contradicting $t_2 \succ_{s_1} t_1$. Hence $s_2 = DA\text{-}STB_{t_2}(R)$ is a school.

By the choice of k, teacher t_2 has not previously been rejected by an application containing s_2. Hence, since t_2 is assigned to s_1 at Step k, she cannot strictly prefer s_2 to s_1. Moreover, she cannot strictly prefer s_1 to s_2: otherwise, since $t_2 \succ_{s_1} t_1$ and t_1 is assigned to s_1 under DA-STB, the DA-STB outcome would violate fairness. Thus $s_1 \; I_{t_2} \; s_2$. Since DA-STB breaks ties according to $\rhd$, and since $t_2 \succ_{s_1} t_1$ while t_1 is assigned to s_1 under DA-STB, it must be that $s_2 \rhd s_1$.

We now continue recursively. Suppose that, for some $i \geq 2$, there exist teachers $(t_1, \ldots, t_i)$ and schools $(s_1, \ldots, s_i)$ such that

i. $DA\text{-}STB_{t_j}(R) = s_j$ for each $j \in \{1, \ldots, i\}$,

ii. $C^{\rhd}_{t_{j+1}}(A^k) = s_j$ for each $j \in \{1, \ldots, i-1\}$,

iii. $s_j \; I_{t_{j+1}} \; s_{j+1}$ for each $j \in \{1, \ldots, i-1\}$, and

iv. $s_{j+1} \rhd s_j$ for each $j \in \{1, \ldots, i-1\}$.

The construction above establishes these properties for $i = 2$.

All seats of s_i must be assigned under $C^{\rhd}(A^k)$. Otherwise, starting from $C^{\rhd}(A^k)$, assign t_1 to s_1, assign t_j to s_j for each $j \in \{2, \ldots, i\}$, and leave all other assignments unchanged. This is feasible: teacher t_1 applies to a set containing s_1, and for each $j \geq 2$, teacher t_j applies to a set containing s_{j-1} and is indifferent between s_{j-1} and s_j, so $s_j \in A^k(t_j)$. The resulting matching assigns one more teacher than $C^{\rhd}(A^k)$, contradicting maximum-size.

Therefore s_i is full under $C^{\rhd}(A^k)$. Since t_i is assigned to s_i under DA-STB but not under $C^{\rhd}(A^k)$, there exists a teacher t_{i+1} such that $C^{\rhd}_{t_{i+1}}(A^k) = s_i$ and $DA\text{-}STB_{t_{i+1}}(R) \neq s_i$.

If $DA\text{-}STB_{t_{i+1}}(R) = \emptyset$, then fairness of DA-STB implies $t_i \succ_{s_i} t_{i+1}$, since t_{i+1} finds s_i acceptable and t_i is assigned to s_i under DA-STB. Construct a matching μ from $C^{\rhd}(A^k)$ by assigning t_j to s_j for each $j \in \{1, \ldots, i\}$, leaving t_{i+1} unassigned, and leaving all other assignments unchanged.

This matching is feasible and has the same size as $C^{\rhd}(A^k)$. Since $s_i \rhd s_{i-1} \rhd \cdots \rhd s_1$, the highest $\vdash$-order pair in the symmetric difference between μ and $C^{\rhd}(A^k)$ is at school s_i; there, μ contains (t_i, s_i) while $C^{\rhd}(A^k)$ contains (t_{i+1}, s_i), and $t_i \succ_{s_i} t_{i+1}$. Hence μ is lexicographically preferred to $C^{\rhd}(A^k)$ among maximum-size matchings, contradicting the definition of $C^{\rhd}$.

It remains to consider the case in which $s_{i+1} = DA\text{-}STB_{t_{i+1}}(R) \in S$. By the choice of k, teacher t_{i+1} has not previously been rejected by an application containing s_{i+1}. Hence, since t_{i+1} is assigned to s_i at Step k, she cannot strictly prefer s_{i+1} to s_i. If $t_i \succ_{s_i} t_{i+1}$, the same switching argument as in the previous paragraph gives a maximum-size matching lexicographically preferred to $C^{\rhd}(A^k)$, again a contradiction. Therefore $t_i \not\succ_{s_i} t_{i+1}$.

Teacher t_{i+1} cannot strictly prefer s_i to s_{i+1}, because then, since t_i is assigned to s_i under DA-STB, fairness of DA-STB would imply $t_i \succ_{s_i} t_{i+1}$, contrary to the previous paragraph. Thus $s_i \; I_{t_{i+1}} \; s_{i+1}$. Moreover, since DA-STB breaks ties according to $\rhd$, if $s_i \rhd s_{i+1}$ then t_{i+1} would strictly prefer s_i to s_{i+1} under the tie-broken preferences, and fairness of DA-STB would again imply $t_i \succ_{s_i} t_{i+1}$. Hence $s_{i+1} \rhd s_i$.

Thus the chain can be extended unless we have already obtained a contradiction. But the chain cannot be extended indefinitely, because each extension produces a strictly higher school in the fixed order $\rhd$: $s_{i+1} \rhd s_i \rhd \cdots \rhd s_1$. Since the set of schools is finite, the construction must eventually reach one of the contradictory cases above. Thus no teacher is ever rejected by an application containing her DA-STB assignment in the DA-CC algorithm. Therefore each teacher weakly prefers her DA-CC assignment to her DA-STB assignment. The improvement is strict for some problem: see Example 6. Hence the DA-CC mechanism induced by $C^{\rhd}$ Pareto improves over DA-STB.

A.7 Proof of Theorem 3

The proof has three parts. First, we show that HC is maximum-size, priority-respecting, and substitutable. This part uses standard augmenting-path arguments for maximum matchings, adapted to the hierarchical structure of application sets and to the critical/noncritical distinction in the definition of HC, which we provide the details in Supplemental Appendix B.4. We now show that DA-HC is fair, strategy-proof, and Pareto improves over DA-STB.

DA-HC is fair and strategy-proof.

Since HC is maximum-size, priority-respecting, and substitutable, then in any outside-option environment with the same schools, capacities, priorities, and geographic hierarchy, the DA-CC mechanism induced by HC is fair and strategy-proof by Theorem 1.

We use this result to rule out violations in the reassignment environment. Fix the primitives of the reassignment environment other than teacher preferences: the set of teachers, the set of schools, capacities, priorities, the geographic hierarchy, and the endowment profile. Suppose, towards a contradiction, that DA-HC violates fairness or strategy-proofness for some preference profile in the corresponding reassignment domain.

Construct an auxiliary outside-option environment with the same teachers, schools, capacities, priorities, and geographic hierarchy as the reassignment environment, and add a common outside option $\emptyset$. Each teacher has the same ranking over schools as in the reassignment problem and ranks the outside option strictly below her endowment school. Thus, the DA-CC mechanism induced by HC is well-defined in this auxiliary environment.

By the definition of DA-HC, for every reported preference profile, the sequence of applications to schools and the choices made by HC in the reassignment problem coincide with those in the auxiliary outside-option environment until the auxiliary outside option would be reached. If a teacher reaches her endowment school ω_t, then she cannot be rejected from it: since HC is priority-respecting and t is ranked among the top q_{ω_t} teachers at ω_t, there cannot be q_{ω_t} teachers assigned to ω_t with higher priority than t. Hence the outside option is never reached in the auxiliary environment, and the DA-CC outcome in the auxiliary outside-option problem coincides with the DA-HC outcome in the reassignment problem.

Therefore, any fairness violation or profitable manipulation for DA-HC in the reassignment environment would induce the corresponding violation for the DA-CC mechanism induced by HC in the auxiliary outside-option environment. This contradicts Theorem 1. Hence DA-HC is fair and strategy-proof.

DA-HC Pareto improves over DA-STB.

Consider an arbitrary problem R. We claim that no teacher is ever rejected by her DA-STB assignment at R in the course of the DA-HC algorithm at R. Towards a contradiction, suppose this is not true. Let k be the earliest step of the DA-HC algorithm at R at which a teacher, say $t \in T$, is rejected by her DA-STB assignment, say $s \in S$. Then, there exists a step, say Step p, of the HC algorithm that runs in Step k of the DA-HC algorithm at R where t is rejected by s. Without loss of generality, assume that before Step p of this run of HC, no teacher is rejected by her DA-STB assignment at R.

Since t is rejected by s, all seats of s must be assigned to higher $\succ_s$-priority teachers at Step p of this run of HC. Moreover, at least one of those teachers, say $t' \in T$ with $t' \succ_s t$, must be assigned to a different school than s, say $s' \in S$, in the DA-STB assignment at R. By our earliest-step assumptions, t' is not rejected by s' either before Step k of the DA-HC algorithm at R or before Step p of this run of HC. Therefore, t' cannot strictly prefer s' to s. Moreover, t' cannot strictly prefer s to s', because DA-STB is fair, $t' \succ_s t$, and t is assigned to s under DA-STB. Hence t' is indifferent between s and s'. In particular, in this run of HC, teacher t' has an application to a region in ϕ that includes both s and s'.

Case 1: s' has a lower index than s. Since s' has a lower index than s and t' has not been rejected by s' before Step p, teacher t' could be assigned to s at Step p only if she is critical and s is the smallest-index school in her application with a vacant seat. But then Lemma 6 implies that no teacher is rejected by s in this run of HC once t' is assigned to s. This contradicts the fact that t is rejected by s at Step p.

Case 2: s' has a higher index than s. Then, in the DA-STB algorithm at R, teacher t' applies to s before applying to s'. Since t' is assigned to s' under DA-STB, she must have been rejected by s at some step of DA-STB. From that step onward, all seats at s are held by teachers with higher priority than t' at s. This contradicts the fact that t is assigned to s under DA-STB while $t' \succ_s t$.

Therefore, no teacher is ever rejected by her DA-STB assignment during DA-HC. It follows that every teacher weakly prefers her DA-HC assignment to her DA-STB assignment. The improvement is strict for some problem: see Example 6. Hence DA-HC Pareto improves over DA-STB.

A.8 Proof of Corollary 1

Suppose, towards a contradiction, that there exists a fair and strategy-proof mechanism φ satisfying the invariance property and Pareto improving over DA-HC. Then there exists a reassignment problem R such that $\varphi(R)$ Pareto dominates $DA\text{-}HC(R)$. Hence, for some teacher $t \in T$, we have

$$\varphi_t(R) = x \; P_t \; y = DA\text{-}HC_t(R).$$

Let $\widehat{R}_t$ be a preference relation such that x is the unique top-ranked school, followed by the endowment school ω_t, and every other school is ranked below ω_t. Let $\widehat{R} = (\widehat{R}_t, R_{-t})$.

We first claim that $\varphi_t(\widehat{R}) = x$. If not, then when teacher t's true preference is $\widehat{R}_t$, she could report R_t and obtain $\varphi_t(R) = x$, her unique top choice, contradicting strategy-proofness of φ.

We next claim that $DA\text{-}HC_t(\widehat{R}) = \omega_t$. If $DA\text{-}HC_t(\widehat{R}) = x$, then when teacher t's true preference is R_t, she could report $\widehat{R}_t$ and obtain x instead of y, contradicting strategy-proofness of DA-HC, since $x \; P_t \; y$. Since, under $\widehat{R}_t$, the only schools weakly preferred to ω_t are x and ω_t, and since teacher t cannot be rejected from her endowment school ω_t, it follows that $DA\text{-}HC_t(\widehat{R}) = \omega_t$.

By the enlargement assumption underlying the invariance property, we may add a new school

z with positive capacity and some priority ordering, and extend preferences so that every teacher ranks z strictly below her endowment school. Since both DA-HC and φ satisfy invariance, both mechanisms assign every teacher to the same original school as at $\widehat{R}$. In particular,

$$\varphi_t = x \quad \text{and} \quad DA\text{-}HC_t = \omega_t$$

in this enlarged problem.

Now modify only teacher t's preference by moving z immediately below x and strictly above ω_t, leaving all other rankings unchanged. Denote the resulting problem by $\widetilde{R}$. Thus, under $\widetilde{R}_t$,

$$x \ P_t \ z \ P_t \ \omega_t,$$

and all schools other than x and z are ranked below ω_t.

We claim that

$$DA\text{-}HC_t(\widetilde{R}) = z.$$

First, $DA\text{-}HC_t(\widetilde{R}) \neq x$. Otherwise, when teacher t's true preference is her enlarged-problem preference with z ranked below ω_t, she could report $\widetilde{R}_t$ and obtain x, her unique top choice, contradicting strategy-proofness of DA-HC. Therefore, under DA-HC at $\widetilde{R}$, teacher t is not assigned to x. Since z is her next acceptable school, z has positive capacity, and every other teacher ranks z strictly below her endowment school, no other teacher applies to z. Hence teacher t is accepted at z.

We next claim that

$$\varphi_t(\widetilde{R}) = x.$$

If not, then when teacher t's true preference is $\widetilde{R}_t$, she could report her previous enlarged-problem preference, under which z is ranked below ω_t, and obtain x, her unique top choice. This would contradict strategy-proofness of φ.

Since φ Pareto improves over DA-HC, $\varphi(\widetilde{R})$ Pareto dominates $DA\text{-}HC(\widetilde{R})$. Moreover, teacher t is assigned z under DA-HC and x under φ, with $x \ P_t \ z$.

Finally, reinterpret the added school z as the outside option. In the corresponding outside-option environment, the reassignment DA-HC outcome at $\widetilde{R}$, after identifying z with the outside option, coincides with the DA-CC outcome induced by HC. Indeed, in the reassignment problem teacher t applies first to x and, after not being assigned to x, is assigned to z; in the outside-option environment this is identified with receiving the outside option. Every other teacher ranks z strictly below her endowment school. Hence no other teacher reaches z before reaching her endowment; and if a teacher reaches her endowment, she cannot be rejected from it because she is ranked among the top q_{ω_j} teachers at ω_j. Therefore the same applications to original schools are processed by HC, and the same choices are made.

The matching $\varphi(\widetilde{R})$ is fair in this outside-option environment. Indeed, $\varphi(\widetilde{R})$ is fair in the reassignment environment, so individual rationality in the reassignment environment implies that every teacher is assigned a school weakly preferred to her endowment. Therefore, any school ranked below a teacher's endowment cannot be strictly preferred to her assignment under $\varphi(\widetilde{R})$, and reinterpreting z as the outside option creates no new individual-rationality, non-wastefulness, or justified-envy violation.

Thus, in the outside-option environment, there exists a fair matching, $\varphi(\widetilde{R})$, that Pareto dominates and size dominates the DA-CC outcome induced by HC: teacher t is moved from the outside option to the acceptable school x, and no teacher is worse off. This contradicts the constrained Pareto-size efficiency of the DA-CC mechanism induced by HC, established in the proof of Theorem 1. Therefore, no fair and strategy-proof mechanism satisfying the invariance property Pareto improves over DA-HC.

A.9 Simulations

We simulate an economy with 250 teachers and 280 schools. Each school has one vacant seat. There are 5 provinces, 35 districts and 140 municipalities. Within each municipality there are 2 schools. Each province contains 7 districts, and each district 4 municipalities.

For each teacher t and each item i we assume that teacher preferences are derived from the following utility function:

$$U_t(i) = \underbrace{\rho \cdot V(i)}_{\text{common utility}} + \underbrace{(1-\rho) \cdot V_t(i)}_{\text{idiosyncratic utility}} \tag{1}$$

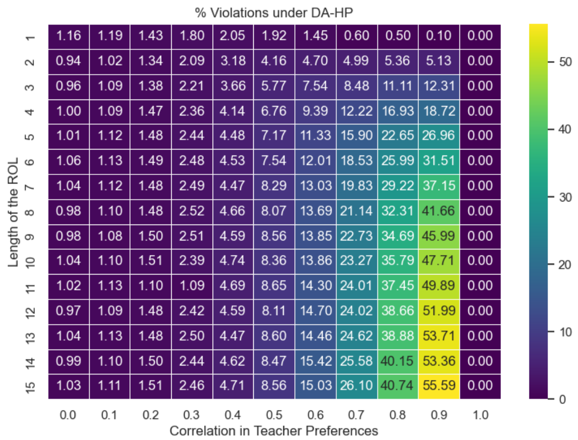
where ρ is a parameter that we make vary between 0 and 1, and $V(\cdot)$ are drawn from iid standard normal distributions, with mean 0 and variance 1. The first term of the utility reflects teachers' common evaluation of the item, and it only varies by item. The higher this term, the more correlated are teachers' preferences. The second term reflects an idiosyncratic preference, and varies between the pair of teacher-item.

For each school, priorities are determined following the current rules in the Italian teacher assignment. [33] This reflects the fact that in the Italian teacher assignment priority rules are common knowledge.

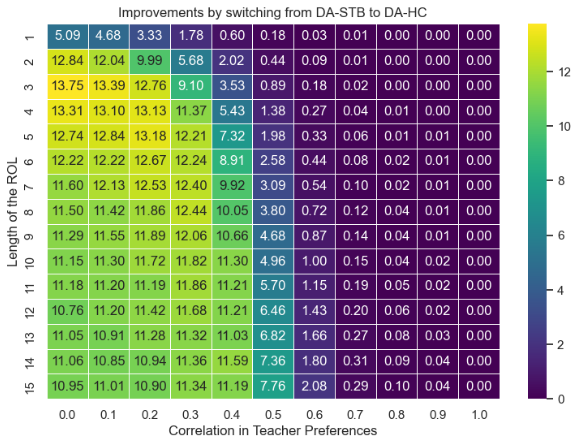
[33] We randomly endow each teacher with a score and a school where they have ownership rights. From these elements we determine teachers' priorities at each school.

We run 1,000 simulations varying teachers' preference correlation parameter and the length of the ROL and report the results in Figure 4. The percentage of teachers with priority violations under DA-HP can be as high as more than half of the teachers and falls to zero when teacher preferences are perfectly aligned. In fact, in this case, DA-HP and DA-HC mechanisms coincide. The percentage of teachers improving by switching to DA-HC from the benchmark DA-STB can be as high as 14% of the teachers. The percentage decreases as the correlation increases, and falls to zero when preferences are perfectly aligned (i.e. when the two mechanisms coincide). Figures 4 also compares welfare improvements by switching to DA-HC from the current mechanism, DA-HP, and vice versa. Note that there is no Pareto dominance relation between the two mechanisms, though DA-HP would be generally more efficient. On the other hand, DA-HC eliminates JE and performs better in efficiency terms compared to the benchmark mechanisms that eliminates JE, the DA-STB.

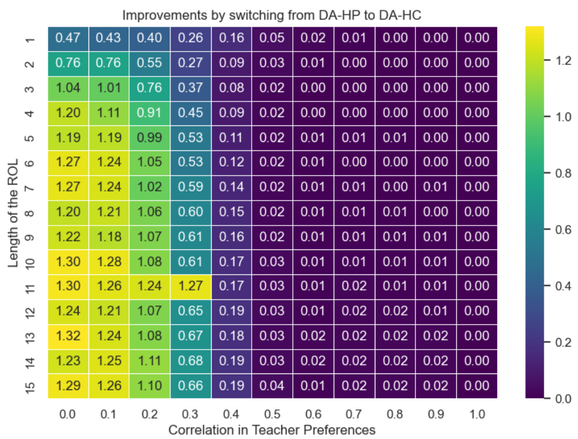
Figure 4: ROLs



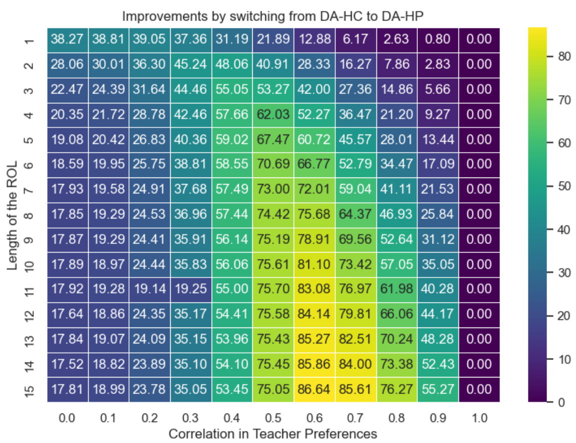
(a) Priority violations under DA-HP



(b) Welfare improvements by switching from DA-STB to DA-HC



(c) Welfare improvements by switching from DA-HP to DA-HC



(d) Welfare improvements by switching from DA-HC to DA-HP

Notes: Panel (a) shows the percentage of teachers whose priority rights are violated under DA-HP, by length of the ROL (y-axis) and correlation of teachers' preferences (x-axis). Panel (b) shows the percentage of teachers improving their welfare by switching from DA-STB to DA-HC, by length of the ROL (y-axis) and correlation of teachers' preferences (x-axis). Panel (c) shows the percentage of teachers improving their welfare by switching from DA-HP to DA-HC, by length of the ROL (y-axis) and correlation of teachers' preferences (x-axis). Panel (d) shows the percentage of teachers improving their welfare by switching from DA-HC to DA-HP, by length of the ROL (y-axis) and correlation of teachers' preferences (x-axis).

Appendix B Supplemental Appendix

B.1 An Overview of Teacher Assignment in Italy

In the Italian public school system, there are around 900,000 teachers (Table 4). Teachers belong to two different categories in terms of their types of contract: tenured teachers with permanent positions and untenured teachers with fixed-term contracts. Approximately 80% of teachers have a tenured position (Table 5).

The public teacher labor market in Italy is mainly organized as a centralized market. The recruitment process of tenured teachers involves the achievement of a qualifying certification and a national-level examination.[34] Wages and salary scale are fixed and determined at the state level, such that there is no bargaining at the individual level.

The assignment to school positions is conducted in a nationwide centralized matching procedure. The assignment to entry-level positions and the reassignment of teaching positions to tenured teachers are both conducted through (distinct) centralized matching procedures. While the two assignment mechanisms share some similarities (for example the possibility to list both single schools and regions), in this paper we focus on the reassignment of tenured teachers.

The current reassignment system for tenured teachers is regulated by a national collective contract which is the result of a national collective bargaining agreement between the government and teachers' unions (*Contratto Collettivo Nazionale di Lavoro*, containing the general principles, integrated by specific rules in the *Contratto Collettivo Nazionale Integrativo*). Details on the implementing rules are then enacted in a Ministerial Decree (*Ordinanza sulla mobilità personale docente, educativo ed ATA*). These rules are periodically revised.

Each year, 10-15% of the tenured teachers apply to be reallocated to a more desirable position, however 40% of them fail to do so and remain in their current position (Table 7). The application period is generally around March-April, where teachers have approximately 15-25 days to submit their application(s), and the outcome of the mechanism is announced around the end of May-June. Teachers can submit their applications online, through the website of the Ministry of Education (under the section *Istanze on line*), and they are also allowed to modify their choices before the deadline.

Teachers can move "geographically", namely from one school to another, or "professionally", between types of school or fields (Table 1). With the geographical mobility teachers can move (i) within the same municipality, (ii) within the same province but between different municipalities, or (iii) between different provinces. With the professional mobility teachers can move (i) between

[34] Similarly, the recruitment of untenured teachers is also centralized, at the province level (for instance, *Graduatorie provinciali di supplenza*), though there have been some attempts to decentralize this process (such as the possibility of a direct contact with school principals, *"chiamata diretta" dei Dirigenti scolastici*).

different types of school (for instance, from a preschool to a primary school), or (ii) between different fields (for instance, a *Biology* teacher can move in a position for teaching *Maths*). The same teacher can apply to both geographic mobility and professional mobility. Moreover, teachers can ask both normal and special educational needs (SEN) teaching positions, indicating a preference order between the two types of positions.

Approximately 80-85% of the teachers apply for geographic mobility. There are no application restrictions to the type of geographic mobility. The same teacher, in the same application, can list destinations within the same municipality of their initial assignment, between different municipalities within the same province of their initial assignment, or between different provinces.

While it eventually amounts to the DA-HP mechanism, the assignment process is explained in the official documents through three phases:[35]

- Phase 1: Geographical transfers within the same municipality of the ownership school
- Phase 2: Geographical transfers within the same province of the ownership school (but between different municipalities)
- Phase 3: Geographical transfers between different provinces, and professional mobility.[36]

The mechanism allocates seats following phases from 1 to 3, which gives rise to what we call "geographical priorities", which are one of the five elements defining school priorities. We used this term to reflect the intrinsic hierarchical geographical structure of the movements within the phases, where teachers moving within the same municipality of their current assignment are given priority over teachers coming from outside the municipality, and teachers moving within the same province of their current assignment are given priority over teachers coming from outside the province. However, within each phase, assignments are determined according to more precise rules, which makes the construction of schools' priority orderings rather complex. In the following section, we provide a description of these rules.[37]

B.1.1 Deferred Acceptance with Hierarchical Priorities

For completeness, we describe the current Italian mechanism, which we refer to as *Deferred Acceptance with Hierarchical Priorities* (DA-HP).

[35]Art.6, paragraph 2 of CCNI

[36]At the end of the second phase, 50% of the remaining vacancies are allocated to new teachers, who are assigned separately within a different matching procedure. Thus, the available vacancies for Round 3 are only half of the vacancies at the end of Round 2. We do not take this detail into account in our model and in the definitions of the mechanisms for the sake of simplicity.

[37]A more exhaustive description can be found in the Appendix of the CCNI.

Fix the official ordering of schools, and identify schools in the model with this ordering. At any step of the algorithm, a school has *vacancy* if the number of applications it has received up to and including that step is strictly smaller than its capacity.

DA-HP Algorithm

Step 1. Each teacher applies to the smallest-index school in her top-ranked acceptable ROL item. Each school first considers applicants whose current ROL item contains no school with vacancy. Among these applicants, the school tentatively accepts the highest-priority teachers up to capacity. If seats remain, the school then considers the remaining applicants and again tentatively accepts the highest-priority teachers up to capacity. Teachers not tentatively accepted by any school are rejected. If no teacher is rejected, the algorithm stops; otherwise proceed to Step 2.

Step $k \geq 2$. Each teacher rejected in Step $k-1$ applies to the smallest-index school in her highest-ranked acceptable ROL item that contains at least one school that has not rejected her before. Teachers not rejected in Step $k-1$ keep their tentative assignments. Each school then considers its tentatively accepted teachers together with its new applicants. It first considers those whose current ROL item contains no school with vacancy, tentatively accepting the highest-priority teachers up to capacity. If seats remain, it then considers the remaining applicants and again tentatively accepts the highest-priority teachers up to capacity. Teachers not tentatively accepted by any school are rejected. If no teacher is rejected, the algorithm stops; otherwise proceed to Step $k+1$.

B.1.2 Preliminary operations

Some operations are conducted before the main phases of the mechanism, because they are related to particular circumstances that deserve special consideration. Consequently, the assignment of these positions has a priority over any other one. We report a detailed list and a brief explanation for each of them:

1. For organizational reasons, some schools can be subject to aggregation and merging with other schools, or suppression, in order to have an “optimal" population of students. Therefore, each year some teachers can be moved to a new school complex because of these reorganizations. In the next year, these teachers are allowed to be reallocated with priority into the new school building where they have been moved in due course.

2. Some teachers can be temporarily be assigned to a different role (*fuori ruolo*). These teachers have the right to be reallocated with priority into their original position within the next 5 years.

3. Teachers can be temporarily assigned to a position (*Docenti in Utilizzazione*). If they have been moved for at least 2 years to a teaching position in prison schools, they have the right to be allocated with priority to this position.

4. The Ministry of Education, at the request of the Department of Public Security, may allow the transfer or the temporary assignment (even in another province) of teachers subject to special security measures, like protection in case of gender-based violence or for exceptional reasons of personal safety.

5. Only for the school year 2019-20, the transfers of teachers to the new fields in “Music High Schools" (*Licei Musicali*) are prioritized.

6. Some tenured teachers currently employed in a specific school-type but, who in the past had a position in another school level of education, can apply to return to their original role.

7. Teachers who had been forced to move into another position because of the suppression of their position can be reallocated to their original position, if the position becomes available again (*Rettifica di titolarità*).

B.1.3 Geographical transfers within the same municipality (Phase 1)

This phase is made by as many movements as municipalities, and concerns all transfers within the same municipality of the current position of the teacher. Movements follow this order:

1. Only in the primary school: transfers between positions (English or regular teaching positions) in the same school complex (*circolo o istituto comprensivo di titolarità*).

2. Transfers of teachers who benefit from the priorities given by the law because of disability and particular health conditions (*First point, art. 13 CCNI*). For these teachers it does not matter whether they come from the initial municipality or not. This type of transfers include any between- or within-province geographical transfer.

3. Transfers of those who have previously been moved, in the last eight years, not voluntarily but because forced by the law, and who ask to move again to their previous position, in the original school or school complex (*Second point, art. 13 CCNI*).

4. Only for high schools, transfers in the same school between daily and evening teaching positions.

5. Transfers of teachers with special priorities due to disability and need of continuous care (*Third point, art. 13 CCNI*).

6. Transfers of teachers with special priorities due to assistance to a child with disability (*Fourth point, art. 13 CCNI*). This applies to the case of municipalities with more districts (i.e. only to municipalities that are metropolitan cities).

7. Transfers of teachers with special priorities due to assistance to a spouse or parent with a disability (*Fourth point, art. 13 CCNI*). This applies to the case of municipalities with more districts (i.e. only to municipalities that are metropolitan cities).

8. Transfers of teachers with priorities given by: 1) at least three years of teaching in hospital or prison schools; 2) at least three years of teaching in adult education or evening courses.

9. Other transfers.

10. Transfers of teachers who must move obligatorily, but have not submitted an application, or they have submitted it, but were not reassigned.

11. Transfers of teachers who have been moved by law, in the last eight years, and asked to move again to their previous municipality (*Fifth point, art. 13 CCNI*).

B.1.4 Geographical transfers within the same province but between different municipalities (Phase 2)

This phase concerns all transfers within the same province of the current position but between different municipalities. Movements follow this order:

1. Transfers of teachers moved by law, but have not submitted any application, or they have submitted it, but were not reassigned yet. The assignment is made considering, among the available positions, the nearest to the previous position of the teacher (i.e. the school where they have their ownership)

2. Transfers of teachers with a disability needing long-term care (*Third point, art. 13 CCNI*).

3. Transfers of teachers asking to move to provide care to a child or someone for whom they have a legal custody (*Fourth point, art. 13 CCNI*).

4. Transfers of teachers asking to move to provide care to the spouse, or a parent with disability (*Fourth point, art. 13 CCNI*).
5. Transfers of teachers with at least three years of teaching in hospital or prison schools.
6. Transfers of teachers with at least three years of teaching in adult education or evening courses.
7. Transfers of teachers whose spouse serves in the Army (*Sixth point, art. 13 CCNI*).
8. Transfers of teachers holding a public office in the local administration (*Seventh point, art. 13 CCNI*).
9. All transfers by teachers who have their school ownership in the province.
10. All teachers (without any priority) who ask to transfer from a special education teaching position to a normal education teaching position (even for transfers in the same municipality).
11. Transfers of teachers who must move obligatorily, and have not been reassigned in previous rounds.
12. Voluntary transfers from teachers whose initial position is in a province subject to administrative changes (*Art. 18 bis CCNI*).

B.1.5 Geographical transfers between different provinces and professional mobility (Phase 3)

This phase concerns all geographical transfers between different provinces and professional mobility (transfers between different fields or type of schools). Movements follow this order:

1. Within or between province transfers across different fields for teachers who benefit from priorities because of disability and particular health conditions (First point, art. 13 CCNI).
2. Within or between province transfers across different types of schools for teachers who benefit from priorities because of disability and particular health conditions (*First point, art. 13 CCNI*).
3. Transfers across different fields for teachers whose field has been suppressed.
4. Transfers across different types of schools for teachers whose field has been suppressed.
5. Transfers across different fields for teachers who in the previous year have taught in a different field than their entitlement.

6. Transfers across different types of schools for teachers who in the previous year have taught in a different field than their entitlement.

7. Transfers across different fields for teachers who do not benefit from any priority.

8. Transfers across different types of schools for teachers who do not benefit from any priority.

9. Between provinces transfers for teachers with a disability who need long-term care (*Third point, art. 13 CCNI*).

10. Between provinces transfers for teachers who ask to move to provide care to a child or someone for whom they have a legal custody with a disability (*Fourth point, art. 13 CCNI*).

11. Between provinces transfers for teachers who ask to move to provide care to the spouse (*Fourth point, art. 13 CCNI*).

12. Between provinces transfers for teachers whose spouse serves in the Army (*Sixth point, art. 13 CCNI*).

13. Transfers for teachers who hold a public office in the local administration (*Seventh point, art. 13 CCNI*).

14. Transfers for teachers who resume their duty, after a trade union leave (*Eighth point, art. 13 CCNI*).

15. Transfers of teachers with at least three years of teaching in hospital or prison schools; transfers of teachers with at least three years of teaching in adult education or evening courses.

16. Between provinces transfers for teachers who do not benefit from any priority.

17. Mandatory transfers for new teachers in 2018-19 that have not obtained an ownership position.

B.1.6 Construction of school priorities

We define schools as the entire school-complex. This definition allows us to refer only to the schools that can be listed by teachers in their application (*Plesso sede di organico*). In fact, teachers cannot list each specific institution within the entire school-complex, but only the principal school-complex that is specifically designated as listable.[38]

[38] Art. 9, CCNI. To identify them, we use specific school denominations as in the Official List published by the Ministry of Education (*Bollettini Ufficiali*).

We consider four levels of education: preschools, primary schools, middle schools and high schools. High schools can be classified into three broad categories: academic high schools (*Licei*), technical high schools (*Istituti Tecnici*), and vocational high schools (*Istituti Professionali*).

For each level of education we distinguish the respective teaching fields. In fact, teachers can request a position only if they have a specific qualification to teach in that field. Thus, the unit of analysis that we consider is the pair school-type. For preschools we consider 10 types, for middle school 43 types, for high school 156 types.

B.2 Teachers' scores

In this section, we report a detailed list of criteria providing scores in geographical and professional mobility.

Table 2: Geographical Mobility

[redacted]	*Score*
Seniority	
A) For each year of teaching	6
A1) For each year of teaching in schools in small islands (in addition to the scores in A)	6
B) For each year of teaching before the nomination in the permanent position	
for voluntary mobility	6
for mandatory mobility	3
B1) For each year of teaching before the nomination in the permanent position	
in a pre-school in small islands (in addition to the scores in B)	
for voluntary mobility	6
for mandatory mobility	3
B2) Only for primary school teachers: for each year of teaching	
in the position of a special teacher in foreign languages	
(for teachers who *only* teach the foreign language)	
from the school year 1992-93 to the school year 1997-98	
if the teaching happened in the school of nomination	0.5
if the teaching happened outside of the school of nomination	1
C) For three consecutive years of teaching, in the role of permanent teacher,	
in the current school (or analogous definitions)	6
For each further year, within five years	2
For each further year, beyond five years	3
For cases in letter C), scores are counted twice if teaching in small islands	
C1) Only for primary school teachers: for three consecutive years of teaching	
in the position of a special teacher in foreign languages	
(for teachers who *also* teach the foreign language)	
from the school year 1992-93 to the school year 1997-98	

Table 2 Continued: Geographical Mobility

(in addition to the scores in A, A1, B, B2, C)	1.5
Only for primary school teachers: for three consecutive years of teaching	
in the position of a special teacher in foreign languages	
(for teachers who *only* teach the foreign language)	
from the school year 1992-93 to the school year 1997-98	
(in addition to the scores in A, A1, B, B2, C)	3
D) Teachers who, for three years, from the school year 2000-01 to the school year 2007-08,	
have not requested a movement between provinces have an additional one-off score of	10
Family Reasons	
A) For reunification to the spouse, or to the parents or to the children	6
B) For each child of age less than 6	4
C) For each child of age greater than 6 and less than 18,	3
or if greater than 18 when they are unable to work	
D) For the assistance of a child with a physical or psychiatric disability,	6
or addicted to drugs, or the assistance of the spouse or a parent unable to work	
Education and Qualifications	
A) For passing a specific public competitive examination for teaching	12
based on qualifications and exams	
B) For each postgraduate qualification or specialization	5
C) For each university degree beyond the required title to teach in the requested role	3
D) For each advanced course with a duration of at least 1 year	1
E) For each four-year degree or master degree or second-level academic degree	5
beyond the required title to teach in the requested role	
F) For holding a PhD degree	5
G) For primary education only, for each training-advanced course	1
in linguistics and language teaching	
H) For each participation, before the school year 2000-01, in the examination board	1
of a high school graduation examination	
I) For each advanced course for content and language integrated learning to teach	1
a non-linguistic subject in a foreign language (with a requirement at the C1 CEFR level)	
L) For each advanced course for content and language integrated learning	0.5
with no requirement at the C1 CEFR level	
Scores from qualifications in B), C), D), E), F), G), I), and L)	
can be cumulated until a maximum of	10

Table 3: Professional Mobility

Description	*Score*

Table 3 Continued: Professional Mobility

Seniority	
A) For each year of teaching	6
A1) For each year of teaching in schools in small islands (in addition to the scores in A)	6
B) For each year of teaching before the nomination in the permanent position	
and for each year of teaching before the nomination in the permanent position	
in the pre-school	6
B1) For each year of teaching before the nomination in the permanent position	
in a pre-school in small islands (in addition to the scores in B)	6
B2) Only for primary school teachers: for each year of teaching	
in the position of a special teacher in foreign languages	
(for teachers who *only* teach the foreign language)	
from the school year 1992-93 to the school year 1997-98	
if the teaching happened in the school of nomination	0.5
if the teaching happened outside of the school of nomination	1
C) For three consecutive years of teaching, in the role of permanent teacher,	
in the current school (or analogous definitions)	6
For each further year, within five years	2
For each further year, beyond five years	3
For cases in letter C), scores are counted twice if teaching in small islands	
C1) Only for primary school teachers: for three consecutive years of teaching	
in the position of a special teacher in foreign languages	
(for teachers who *also* teach the foreign language)	
from the school year 1992-93 to the school year 1997-98	
(in addition to the scores in A, A1, B, B2, C)	1.5
Only for primary school teachers: for three consecutive years of teaching	
in the position of a special teacher in foreign languages	
(for teachers who *only* teach the foreign language)	
from the school year 1992-93 to the school year 1997-98	
(in addition to the scores in A, A1, B, B2, C)	3
D) Teachers who, for three years, from the school year 2000-01 to the school year 2007-08,	
have not requested a movement between provinces have an additional one-off score of	10
Education and Qualifications	
A) For passing a specific public competitive examination	
based on qualifications and exams for teaching	
in the current role or a higher role	12
B) For passing other public competitive examinations	
based on qualifications and exams for teaching	
in roles of the same level or higher than the current role	6
C) For each postgraduate qualification or specialization	5
D) For each university degree beyond the required title to teach in the requested role	3

Table 3 Continued: Professional Mobility

E) For each advanced course with a duration of at least 1 year	1
F) For each four-year degree or master degree or second-level academic degree beyond the required title to teach in the requested role	6
G) For holding a PhD degree	6
H) For primary education only, for each training-advanced course in linguistics and language teaching	1
I) For each participation, before the school year 2000-01, in the examination board of a high school graduation examination	1
L) For each year of teaching (or a period beyond 180 days) in the same role requested	3
M) For each advanced course for content and language integrated learning to teach a non-linguistic subject in a foreign language (with a requirement at the C1 CEFR level)	1
N) For each advanced course for content and language integrated learning with no requirement at the C1 CEFR level	0.5

B.2.1 Special Priorities

There is a system of special priorities, such that teachers who belong to specific categories might gain additional priority. These special priorities are:

1. Disability and particular health conditions. Among these, the highest priority is given in particular to those who have a blindness condition, or need a regular haemodialysis treatment.

2. Teachers who have been moved, because forced by the law, in the last eight years, who ask to move again to their previous position.

3. Teachers with a disability who need long-term care.

4. Teachers who ask to move to provide care to the spouse, or to a child, with a disability; Teachers (identified as the only reference) who ask to move to provide care to a parent with disability; Teachers who move to provide care to someone for whom they have a legal custody.

5. Teachers who have been moved, because forced by the law, in the last eight years, who ask to move to the municipality where there was their previous position.

6. Teachers whose spouse serves in the Army.

Table 4: Total Teachers by Type of School

School Type	*2018-19*	*2019-20*	*2020-21*	*2021-22*	*2022-23*
Preschool	11.58%	11.52%	11.20%	11.13%	11.11%
Primary School	32.03%	32.08%	32.20%	32.31%	32.65%
Middle School	22.41%	22.28%	22.29%	22.19%	22.08%
High School	33.98%	34.12%	34.31%	34.38%	34.16%
Total Teachers	886,175	902,487	907,929	923,854	943,681

Notes: This table reports the total population of teachers in Italy and the corresponding fraction by school type (preschools, primary schools, middle schools, and high schools). The numbers include both tenured and untenured teachers, and both normal education and SEN teachers. We may note a slight increase of the total teachers across years. We report the last 5 years of available data.

Table 5: Teachers by contractual type

Contractual Type	*2018-19*	*2019-20*	*2020-21*	*2021-22*	*2022-23*
Untenured	18.47%	20.61%	23.39%	24.35%	25.60%
Tenured	81.53%	79.39%	76.61%	75.65%	74.40%
Total Teachers	886,175	902,487	907,929	923,854	943,681

Notes: This table reports the fraction of tenured and untenured teachers on the total population of teachers in Italy. We may note a decrease of the fraction of tenured contractual teachers over years. We report the last 5 years of available data.

7. Teachers who hold a public office in the local administration.

8. Teachers who resume their duty, after a trade union leave (regulated by *C.C.N.Q., 4 December 2017*).

B.3 Data and Descriptives

We use administrative data provided by the Italian Ministry of Education on the universe of applications for the teacher reassignment procedure in the school-years 2019-20, 2020-21, and 2021-22. Data include different datasets, which we merge for the analysis.

i. *Teacher applications*. A dataset containing information about the type of the application, the teacher's score and, where applicable, teacher's special priorities.

ii. *Teacher rank order lists*. A dataset containing the rank order list of submitted preferences.

iii. *Teacher ownership rights*. A dataset containing information on the initial assignment of the teacher, namely the teaching position where they have their ownership right (*scuola o*

Table 6: Teachers and applications

	2019-20	*2020-21*	*2021-22*
Teachers Moving	66,296	56,732	48,802
Total Applications	129,803	108,677	87,454
Total Applicants	115,534	96,577	78,232

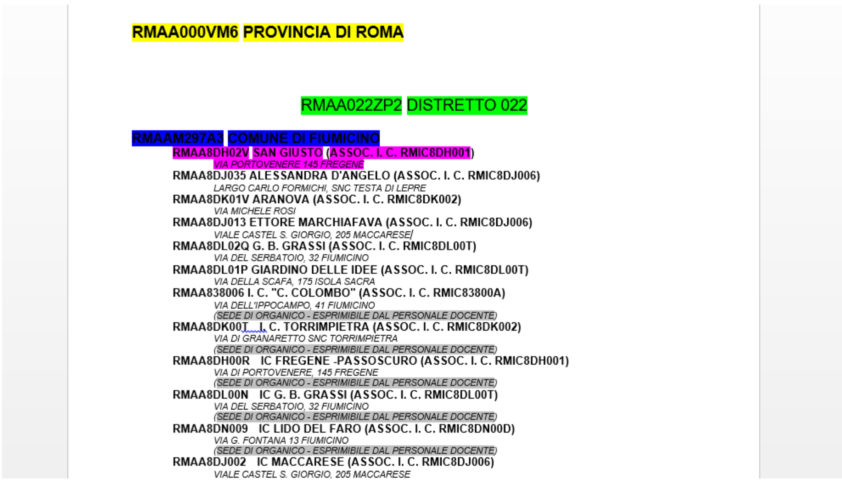
Notes: This table reports the total number of tenured teachers applying for mobility, the number of applications, and the number of teachers whose application is successful. We report data for school years from 2019-20 to 2021-22.

Table 7: Applications by Type

Application Type	*2019-20*	*2020-21*	*2021-22*
Geographical Mobility	83.04%	83.10%	82.14%
Professional Mobility			
Mobility between fields	3.77%	3.67%	3.57%
Mobility between school types	13.19%	13.23%	14.29%
Total Applications	129,803	108,677	87,454

Notes: This table reports the fraction of applications per type of application. We report data for school years from 2019-20 to 2021-22.

Figure 5: Official List of Schools (*Bollettini Ufficiali*)


RMAA000VM6 PROVINCIA DI ROMA

RMAA022ZP2 DISTRETTO 022

RMAAM297A3 COMUNE DI FIUMICINO
RMAA8DH02V SAN GIUSTO (ASSOC. I. C. RMIC8DH001)
VIA PORTOVENERE 145 FREGENE
RMAA8DJ035 ALESSANDRA D'ANGELO (ASSOC. I. C. RMIC8DJ006)
LARGO CARLO FORMICHI, SNC TESTA DI LEPRE
RMAA8DK01V ARANOVA (ASSOC. I. C. RMIC8DK002)
VIA MICHELE ROSI
RMAA8DJ013 ETTORE MARCHIAFAVA (ASSOC. I. C. RMIC8DJ006)
VIALE CASTEL S. GIORGIO, 205 MACCARESE
RMAA8DL02Q G. B. GRASSI (ASSOC. I. C. RMIC8DL00T)
VIA DEL SERBATOIO, 32 FIUMICINO
RMAA8DL01P GIARDINO DELLE IDEE (ASSOC. I. C. RMIC8DL00T)
VIA DELLA SCAFA, 175 ISOLA SACRA
RMAA838006 I. C. "C. COLOMBO" (ASSOC. I. C. RMIC83800A)
VIA DELL'IPPOCAMPO, 41 FIUMICINO
(SEDE DI ORGANICO - ESPRIMIBILE DAL PERSONALE DOCENTE)
RMAA8DK00T I. C. TORRIMPIETRA (ASSOC. I. C. RMIC8DK002)
VIA DI GRANARETTO SNC TORRIMPIETRA
(SEDE DI ORGANICO - ESPRIMIBILE DAL PERSONALE DOCENTE)
RMAA8DH00R IC FREGENE -PASSOSCURO (ASSOC. I. C. RMIC8DH001)
VIA DI PORTOVENERE, 145 FREGENE
(SEDE DI ORGANICO - ESPRIMIBILE DAL PERSONALE DOCENTE)
RMAA8DL00N IC G. B. GRASSI (ASSOC. I. C. RMIC8DL00T)
VIA DEL SERBATOIO, 32 FIUMICINO
(SEDE DI ORGANICO - ESPRIMIBILE DAL PERSONALE DOCENTE)
RMAA8DN009 IC LIDO DEL FARO (ASSOC. I. C. RMIC8DN00D)
VIA G. FONTANA 13 FIUMICINO
(SEDE DI ORGANICO - ESPRIMIBILE DAL PERSONALE DOCENTE)
RMAA8DJ002 IC MACCARESE (ASSOC. I. C. RMIC8DJ006)
VIALE CASTEL S. GIORGIO, 205 MACCARESE

Notes: This is the structure of the Official List of Schools (*Bollettini Ufficiali*), published by the Italian Ministry of Education. Highlighted in yellow, there is the province id and the name of the province (here, Rome). Highlighted in green, there is the district id, and the name of the district (here, District n.22). Highlighted in blue, there is the municipality id, and the name of the municipality (here, Fiumicino). Highlighted in pink, there is the school id, the name of the school, the id of the associated school complex, and the school address. Highlighted in grey, there is the specific reference for schools that can be listed in the teacher application (*Plesso sede di organico*).

Table 8: Teacher Characteristics

	2019-20	*2020-21*	*2021-22*
Family reasons			
Family Reunification	55.83%	57.16%	54.15%
Children's Age < 6	15.57%	15.12%	13.02%
Children's Age < 18	31.14%	32.37%	31.26%
Assistance to Family Member			
Child	1.12%	1.15%	1.19%
Spouse	0.28%	0.31%	0.31%
Parent	1.81%	2.00%	2.31%
Education and Certifications			
PhD	4.01%	3.78%	3.69%
Postgraduate specialization	4.92%	4.93%	5.07%
Advanced course (duration > 1 year)	52.46%	53.77%	52.88%
Seniority			
Average years of tenured teaching	6.81	7.07	8.09
Over 25 years of tenured teaching	4.36%	11.12%	7.43%
Average years of untenured teaching (before tenure)	6.22	6.18	6.06
Special Priorities			
Blindness	0.01%	0.01%	0.01%
Haemodialysis	0.01%	0.01%	0.01%
Teachers moved by law in the last 8 years	3.19%	3.32%	2.84%
Long-term disability	2.11%	2.38%	2.40%
Provide care to family member	3.21%	3.46%	3.80%
Teachers moved by law in the last 8 years, asking to come back to the same municipality	3.28%	3.40%	2.92%
Spouse in the Army	0.37%	0.39%	0.37%
Public office in the local administration	0.22%	0.24%	0.19%
Trade union leave	0.02%	0.01%	0.01%

Notes: This table reports the main teacher characteristics, classified as: family reasons, education and qualifications, seniority, and special priorities. We report data for school years from 2019-20 to 2021-22.

Table 9: Schools in the Hierarchy

	N	Mean school buildings	Min	Max	N	Mean school-complexes	Min	Max
Panel A. Preschools								
Provinces	101	152.36	59	939	101	47.95	9	324
Districts	658	17.86	1	88	652	5.11	1	28
Small Districts	105	14.51	1	68	105	5.01	1	20
Municipalities	5,681	5.09	1	70	2,827	1.93	1	15
Big Municipalities	19	89.42	14	366	19	34.22	5	142
Schools	18,541				4,995			
Panel B. Primary schools								
Provinces	105	124.35	15	709	101	54.48	10	340
Districts	664	18.50	1	293	655	5.54	1	30
Small Districts	105	11.95	3	58	105	5.68	1	21
Municipalities	6,690	3.90	1	51	3,019	2.11	1	19
Big Municipalities	19	85.47	27	393	19	46.00	8	203
Schools	16,240				5,593			
Panel C. Middle schools								
Provinces	105	59.52	5	358	101	50.41	9	326
Districts	663	7.81	1	77	654	5.17	1	30
Small Districts	105	3.81	1	22	654	5.29	1	21
Municipalities	5,234	1.79	1	22	3,025	1.95	1	19
Big Municipalities	19	44.51	8	199	19	43.48	7	193
Schools	7,795				5,276			
Panel D. High schools								
Provinces	105	120.03	9	764	101	8,60	19	466
Districts	654	13.45	1	114	618	8.82	1	41
Small Districts	105	12.704	4	60	105	9.21	1	46
Municipalities	3,160	6.35	1	51	1,441	5.82	1	41
Big Municipalities	19	100.09	24	449	19	59.34	19	259
Schools	13,842				8,596			

Notes: This table reports information on the geographic hierarchical structure.

provincia di titolarità).

iv. *Vacancies*. A dataset containing information about the vacancies available for new teaching positions.

v. *Assignment outcome*. A dataset containing the matching outcome.

vi. *Official Bullettin*. A dataset publicly available from the website of the Ministry of Education containing the Official List of the schools (*Bollettini Ufficiali*), from which we recover the exact order used in the tie-breaking procedure (Figure 5).

vii. *List of the schools*. A dataset containing the universe of the schools, with geographical information on the corresponding municipality, district and province. Since teachers can only indicate the school complex (*Plesso sede di organico*) in their applications,[39] we merge this dataset with the *Official Bullettin* to identify schools that can be listed.

Geographical Hierarchy

The geographical hierarchical structure that is relevant for teacher assignment has four levels: provinces, which include districts, which include municipalities, and which include schools.[40] Italy is currently divided into 20 administrative regions, which in turn contain 107 provinces, which in turn contain 7901 municipalities.[41] However, in the mobility of tenured teachers, administrative regions do not play any role, thus we will not consider them. Differently from the general administrative subdivision, the Ministry of Education considers also another level in the hierarchy, which is represented by the districts.[42] In total there are 769 districts and 6970 relevant municipalities (Table 9).[43]

[39]Teachers cannot list each specific institution, but only the school that is specifically designated as listable, within the entire school complex. See *Art. 9, Modalità di indicazione delle sedi di organico, Contratto Collettivo Nazionale Integrativo (2019/20, 2020/21, 2021/22)*.

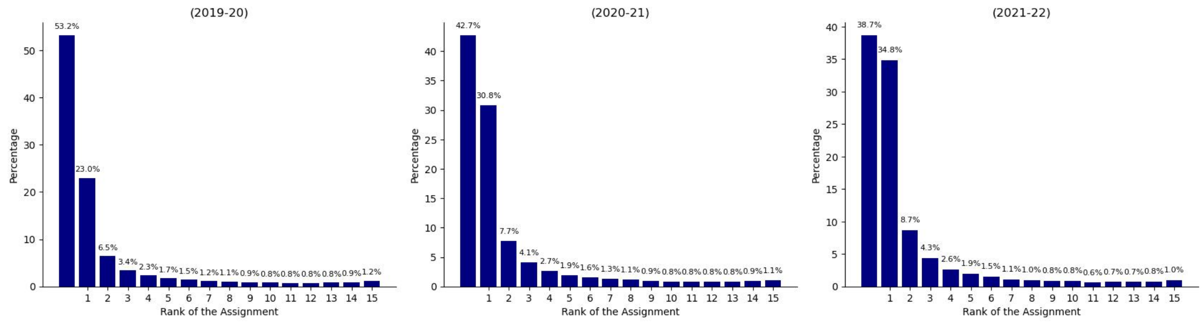
[40]In practice, there are big municipalities, not contained in any district, but containing small districts. In our analysis, big municipalities containing small district within them are Bari, Bologna, Cagliari, Catania, Firenze, Genova, Messina, Milano, Modena, Napoli, Padova, Palermo, Roma, Taranto, Torino, Treviso, Trieste, Venezia, Verona, Vicenza. However, the four level hierarchy is respected also in this case, by simply relabelling municipalities and districts.

[41]In this classification the provinces of Aosta, Fermo, Barletta-Trani-Andria and Sud Sardegna are not considered. At the same time the province of Bolzano is instead divided into three, according to the linguistic minorities: Bolzano (Italian language), Bolzano (Ladin language), Bolzano (German language). For simplicity we consider 101 provinces, excluding Trento, Bolzano (Italian language), Bolzano (Ladin language), Bolzano (German language).

[42]Districts were introduced in 1974 (art. 9 of DPR, n. 416, 31 May 1974) but along the years they have lost any effective role in the organization and administration of the educational network. However, they still play a role in the mobility of teachers, since teachers can submit this territorial entities as preferences.

[43]There can be municipalities without schools, thus this number is lower than the number of total municipalities.

Figure 6: Rank of the Assignment



Notes: These graphs show the position in the submitted rank ordered list for the assigned outcome.

Teacher applications

There were around 100,000 applications each year. Most of them, 40-41%, are from high school teachers, while the remaining 28-29% are from primary school teachers, 18% from middle school teachers, and 12-13% from preschool teachers (Table 4). The majority of the applications are requests for geographic transfers (82-83%).

Approximately 23-35% of the teachers are reallocated to their first stated position (See Figure 6 for the entire assignment rank distribution). Approximately 40-50% of the teachers fail to be reassigned to a more preferable position than their current positions. Most of the teachers tend to reapply across years (Table 10). Some teachers are not allowed to reapply because of a waiting-time constraint.[44]

Vacancies

The number of vacancies at each school is determined by four factors:

1. new teaching positions formed in that year,
2. positions that become vacant (e.g., because the previous teachers retire),
3. positions not assigned to permanent teachers, and
4. positions that become available during the assignment process itself (i.e., some teachers from that school participating in the reassignment system and securing a position in a different school)

[44] In the years we are considering, all teachers listing a school as a singleton school, who are successfully assigned to that school, are subject to a constraint to remain in that position for three years.

Table 10: Reapplications

	2019-20	*2020-21*	*2021-22*
Applicants who apply in 2019-20, *2020-21*, and *2021-22*	36,517 [31.61%]	36,517 [37.81%]	36,517 [46.68%]
Applicants who apply in 2019-20, and *2020-21*	60,329 [52.22%]	60,329 [62.47%]	-
Applicants who apply in 2019-20 and 2021-22	41,519 [35.94%]	-	41,519 [53.07%]
Applicants who apply in 2020-21, and *2021-22*	-	46,720 [48.38%]	46,720 [59.72%]
Applicants	115,534	96,577	78,232
Applications	129,803	108,677	87,454

Notes: This table reports the fraction of applicants reapplying multiple years. We report data for school years from 2019-20 to 2021-22.

Table 11: Vacancies

	2019			2020			2021		
	Mean	Min	Max	Mean	Min	Max	Mean	Min	Max
Preschools									
Schools	1.6	0	32	2.3	0	36	2.4	0	36
Municipalities[a]	2.9	0	130	4.0	0	228	4.2	0	260
Districts	10.9	0	118	15.2	0	154	16.0	0	150
Provinces	80.8	2	528	111.9	12	866	118.1	10	918
Primary Schools									
Schools	4.1	0	58	6.1	0	66	7.8	0	70
Municipalities	8.2	0	800	12.3	0	1610	15.7	0	3004
Districts	32.7	0	284	49.3	0	334	62.8	0	504
Provinces	246.2	0	2596	368.2	36	3992	468.1	22	5320
Middle Schools									
Schools	8.6	0	84	10.2	0	90	13.6	0	104
Municipalities	16.2	0	1770	19.5	0	2232	26.1	0	2996
Districts	65.1	0	392	78.4	0	406	104.3	0	536
Provinces	489.5	52	3950	584.9	82	4676	776.9	110	6076
High Schools									
Schools	13.3	0	148	17.8	0	166	24.3	0	228
Municipalities	46.7	0	1098	63.1	0	1756	86.5	0	3078
Districts	69.2	0	500	94.9	0	596	130.5	0	808
Provinces	474.4	46	2632	647.2	78	3734	890.3	98	5390

Notes: The reported vacancies exclude the potential vacancies arising from positions that become available during the assignment process itself.
[a] Note that municipalities include both small and big municipalities.

Schools' priority orderings

We have constructed schools' priority orderings based on the available information.[45] Schools' priorities are determined roughly by five factors. First, teachers have the highest priority in their current school, such that if they cannot be transferred to another position, then they are guaranteed to be reassigned to their current position (individual rationality constraint). Second, there are "special priorities", for instance, due to specific health conditions and special circumstances.[46] Third, teachers gain priority depending on their "scores", which are calculated by considering seniority, educational and training qualifications, and family reasons.[47] Fourth, there are geographical priorities where a teacher gains priority in the municipality of her current school over applicants from other municipalities, and in the province of her current school over applicants from other provinces. The final priority ordering at each school is a strict ranking of teachers, where in case of ties after applying the above five factors, teacher age is used as a tie-breaker.

B.4 HC is maximum-size, priority-respecting, and substitutable

Let A be an arbitrary (hierarchical) application profile. We will first introduce some notation and prove a set of lemmas and propositions.

Given $T' \subseteq T$, we denote by $A_{T'}$ the application profile obtained from A by keeping $A(t)$ unchanged for each $t \in T'$ and setting $A(t) = \emptyset$ for each $t \notin T'$. For each teacher $t \in T$, we denote by A_{-t} the application profile obtained from A by setting $A(t) = \emptyset$ and leaving all other applications unchanged.

The following lemma is a simple observation that doesn't require a proof, while we will invoke this observation in several proofs.

Lemma 2. *Let (T', T'') be a partition of T and (q', q'') be such that, for each $s \in S$, $q'_s + q''_s = q_s$. Let μ' be a matching for the problem $(A_{T'}, q')$ and μ'' be a matching for the problem $(A_{T''}, q'')$. If there exists a maximum size matching μ for the combined problem (A, q) that coincides with μ' for $A_{T'}$, and if μ'' is a maximum size matching for $(A_{T''}, q'')$, then μ' combined with μ'' is a maximum size matching for the combined problem (A, q).*

We call a pair (A', q') with A' obtained from A by replacing some teachers' applications with $\emptyset$ and $q'_s \leq q_s$ for each $s \in S$, a **reduced problem** if there is a way of assigning teachers whose

[45] Out of more than 40 different criteria that determine priorities (See Section B.1.6), we did not have access to few of those criteria and therefore we simplified the construction of schools' priorities. We expect that these criteria would only apply for a very limited number of teachers and would have negligible effect on the outcomes as far as our analysis is concerned.

[46] Appendix B.2.1 provides a complete list.

[47] See Appendix B.2 for a complete detailed list.

applications were removed to schools that respects their original applications and the capacity profile $q - q'$. We next show that, for any reduced problem, there exists a maximum size matching in which the smallest-index teacher among the finest applications is accepted to the smallest-index school in her application.[48]

Lemma 3. *Let (A', q') be any reduced problem. There exists a maximum size matching in which the smallest-index teacher among finest applications is accepted to the smallest-index school in her application.*

Proof. Given any reduced problem (A', q'), consider the following choice rule, which we call the *simple hierarchical choice rule* (SHC). Note that, in contrast with HC, SHC does not take into account the priority orderings.

SHC

Step 1*:* Consider the smallest-index teacher among finest applications, say t. Accept (permanently) t to the school in $A'(t)$ with the smallest index. Move to the next step.

Steps $k > 1$*:* If there is no vacant seat left or all teachers with an application are considered, then terminate. Otherwise, among the teachers who have not been considered yet, consider the smallest-index teacher among finest applications, say t. Accept (permanently) t to the school in $A'(t)$ with the smallest index that has a vacant seat (if any). Move to the next step.

We will show that, for any reduced problem (A', q'), $SHC(A', q')$ is *maximum-size*. This will also imply the statement of the lemma.

Suppose, towards a contradiction, that there is another matching for the reduced problem that has a greater size than $SHC(A', q')$. Note that for any teacher who is not assigned to any school, none of her applied schools has any vacant seat in $SHC(A', q')$ by construction. Then, there exists a list of teachers $t_1, \ldots, t_k$ with $k \geq 2$, and a school s such that t_1 is not assigned to any school in $SHC(A', q')$, all other teachers in the list are assigned to some schools in $SHC(A', q')$, and s has a vacant seat in $SHC(A', q')$, such that the assigned school of t_i is feasible also for t_{i-1} for each $i \in \{1, \ldots, k\}$ and s is feasible for t_k, i.e., there exists an *augmenting path* (Berge, 1957). Without loss of generality, assume that $t_1, \ldots, t_k$ is a shortest size augmenting path. Note that the assigned school of t_3 is not in $A'(t_1)$ (since otherwise there would be a shorter augmenting path) while it is in $A'(t_2)$. Moreover, the assigned school of t_2 is in $A'(t_1) \cap A'(t_2)$. Hence, t_2 has a strictly

[48] Note that the proof includes a simple polynomial time algorithm that finds a maximum size matching in for a (hierarchical) application profile, while there are already known algorithms in the literature that find a maximum size matching in the general bipartite matching context.

coarser application than t_1, contradicting that t_2 is considered before t_1 in the SHC algorithm since t_2 was assigned a school that is in $A'(t_1) \cap A'(t_2)$ while t_1 was left with no feasible vacant seat in the step where t_1 was considered. $\square$

We next show that, as we iterate the steps of the $HC(A)$ algorithm, the set of critical teachers is preserved. For any set of plausible applications A and any capacity profile q', we say an applicant t is critical in (A, q') if t is critical in A when the capacity profile is q'.

Lemma 4. *For any Step k of the $HC(A)$ algorithm, let (A^k, q^k) denote the reduced problem at the beginning of Step k, i.e., $A^k(t)$ is equal to $A(t)$ for each teacher who is not tentatively accepted by any school and who has not been rejected by all applied schools by the beginning of Step k, and $A^k(t) = \emptyset$ otherwise, and q^k is the profile of vacant seats at the beginning of Step k. Take any teacher t with $A^k(t) \neq \emptyset$. Then, t is critical in (A, q) if and only if t is critical in (A^k, q^k).*

Proof. Suppose, towards a contradiction, that the statement is not true. Without loss of generality, assume that Step k is the earliest step of the algorithm at which the statement is violated. Let t' with $A(t') \neq \emptyset$ be the teacher considered in Step $k-1$.

Case 1: Suppose, towards a contradiction, that a teacher t is critical in (A^{k-1}, q^{k-1}) but not critical in (A^k, q^k).

Subcase 1: t' is tentatively accepted to the smallest index school in $A(t')$ that has a vacant seat in Step $k-1$ of the algorithm, say s. Since t is not critical in (A^k, q^k), there exists a maximum size matching for the reduced problem (A^k, q^k), say μ, in which t is not assigned to any school. By Lemma 3, there exists a maximum size matching for the reduced problem (A^{k-1}, q^{k-1}) in which t' is accepted to s. But then, by Lemma 2, the matching obtained from μ by assigning t' to s is a maximum size matching for the reduced problem (A^{k-1}, q^{k-1}), in which t is not assigned to any school, contradicting that t is critical in (A^{k-1}, q^{k-1}).

Subcase 2: t' replaces some teacher t'' with $A(t'') \neq \emptyset$ in some school s in Step $k-1$ of the algorithm. Note that t' is not critical in (A^{k-1}, q^{k-1}).

Suppose that $A(t'') \subseteq A(t')$. Note that t'' is not critical in (A^k, q^k) since $A(t'') \subseteq A(t')$, $q^{k-1} = q^k$, A^{k-1} is obtained from A^k by replacing the active application of t'' with the active application of t', and t' is not critical in (A^{k-1}, q^{k-1}). Then, the size of the maximum size matching in (A^{k-1}, q^{k-1}) is the same as the size of the maximum size matching in (A^k, q^k). Since t is not critical in (A^k, q^k), there exists a maximum size matching for the reduced problem (A^k, q^k), say μ, in which t is not assigned to any school. Then, the matching obtained from μ by replacing t' with t'' in s is a maximum size matching for the reduced problem (A^{k-1}, q^{k-1}), in which t is not assigned to any school, contradicting that t is critical in (A^{k-1}, q^{k-1}).

Suppose that $A(t') \subsetneq A(t'')$. Observe that there exists a list of teachers $t_1, \ldots, t_n$ with non-empty applications, a list of schools $s_2, \ldots, s_n$ such that

- $t_{n-1} = t'$ and $t_n = t''$,
- $s_n = s$,
- $A(t_j) \subseteq A(t_1)$ for each $j > 1$, and
- t_1 replaces t_2 in s_2 at some Step $k_1 < k$ of the algorithm, then in the next step t_2 replaces t_3 in s_3, ..., then finally $t_{n-1} = t'$ replaces $t_n = t''$ in $s_n = s$ in Step $k-1$ of the algorithm.

Note that t_1 is not critical in (A^{k_1}, q^{k_1}). Also note that t'' is not critical in (A^k, q^k) since $A(t'') \subseteq A(t_1)$, $q^{k_1} = q^k$, A^{k_1} is obtained from A^k by replacing the active application of t'' with the active application of t_1, and t_1 is not critical in (A^{k_1}, q^{k_1}). Then, the size of the maximum size matching in (A^{k_1}, q^{k_1}) is the same as the size of the maximum size matching in (A^k, q^k). Since t is not critical in (A^k, q^k), there exists a maximum size matching for the reduced problem (A^k, q^k), say μ, in which t is not assigned to any school. Then, the matching obtained from μ by assigning $t_n = t''$ to $s_n = s$, $t_{n-1} = t'$ to s_{n-1},..., t_2 to s_2, and leaving t_1 unassigned, is a maximum size matching for the reduced problem (A^{k_1}, q^{k_1}), in which t is not assigned to any school, contradicting that t is critical in (A^{k_1}, q^{k_1}).

Case 2: Suppose, towards a contradiction, that a teacher t is not critical in (A^{k-1}, q^{k-1}) but critical in (A^k, q^k).

Subcase 1: t' is tentatively accepted to the smallest index school in $A(t')$ that has a vacant seat in Step $k-1$ of the algorithm, say s. Since t is not critical in (A^{k-1}, q^{k-1}), there exists a maximum size matching for the reduced problem (A^{k-1}, q^{k-1}), say μ, in which t is not assigned to any school. Without loss of generality, assume that t' is assigned to s in μ (otherwise, either we can move t' to a vacant seat in s if there is any vacant seat, or we can switch the schools of t' and another teacher who is assigned s, which is feasible since t' is a teacher with a finest application). Then, the matching obtained from μ by removing t' is a maximum size matching for the reduced problem (A^k, q^k), in which t is not assigned to any school, contradicting that t is critical in (A^k, q^k).

Subcase 2: t' replaces some teacher t'' in some school s in Step $k-1$ of the algorithm. Note that t' is not critical in (A^{k-1}, q^{k-1}).

Suppose that $A(t'') \subseteq A(t')$. Note that t'' is not critical in (A^k, q^k) since $A(t'') \subseteq A(t')$, $q^{k-1} = q^k$, A^{k-1} is obtained from A^k by replacing the active application of t'' with the active application of t', and t' is not critical in (A^{k-1}, q^{k-1}). Then, the size of the maximum size matching in (A^{k-1}, q^{k-1}) is the same as the size of the maximum size matching in (A^k, q^k). Since t is not critical in (A^{k-1}, q^{k-1}), there exists a maximum size matching for the reduced problem (A^{k-1}, q^{k-1}), say μ, in which t is not assigned to any school. Then, the matching obtained from μ by replacing t'' with t' in s is a maximum size matching for the reduced problem (A^k, q^k), in which t is not assigned to any school, contradicting that t is critical in (A^k, q^k).

Suppose that $A(t') \subsetneq A(t'')$. As above, there exists a list of teachers $t_1, \ldots, t_n$ with non-empty applications, a list of schools $s_2, \ldots, s_n$ such that

- $t_{n-1} = t'$ and $t_n = t''$,
- $s_n = s$,
- $A(t_j) \subseteq A(t_1)$ for each $j > 1$, and
- t_1 replaces t_2 in s_2 at some Step $k_1 < k$ of the algorithm, then in the next step t_2 replaces t_3 in s_3, ..., then finally $t_{n-1} = t'$ replaces $t_n = t''$ in $s_n = s$ in Step $k-1$ of the algorithm.

Note that t_1 is not critical in (A^{k_1}, q^{k_1}). Also note that t'' is not critical in (A^k, q^k) since $A(t'') \subseteq A(t_1)$, $q^{k_1} = q^k$, A^{k_1} is obtained from A^k by replacing the active application of t'' with the active application of t_1, and t_1 is not critical in (A^{k_1}, q^{k_1}). Then, the size of the maximum size matching in (A^{k_1}, q^{k_1}) is the same as the size of the maximum size matching in (A^k, q^k). Since t is not critical in (A^{k-1}, q^{k-1}), there exists a maximum size matching for the reduced problem (A^{k-1}, q^{k-1}), say μ, in which t is not assigned to any school. Then, the matching obtained from μ by assigning t_1 to s_2, t_2 to s_3,..., $t_{n-1} = t'$ to $s_n = s$, and leaving t'' unassigned, is a maximum size matching for the reduced problem (A^k, q^k), in which t is not assigned to any school, contradicting that t is critical in (A^k, q^k). □

Lemma 5. *For any Step k of the $HC(A)$ algorithm, let μ^k be the matching at the end of Step k (determined by the tentative acceptances at the end of Step k). There exists a maximum size matching μ for the entire problem (A, q) which coincides with μ^k for the teachers who are tentatively accepted at the end of Step k.*

Proof. For any Step k of the $HC(A)$ algorithm, let (A^k, q^k) denote the reduced problem at the beginning of Step k, i.e., $A^k(t)$ is equal to $A(t)$ for each teacher who is not tentatively accepted by any school and who has not been rejected by all feasible schools by the beginning of Step k, and $A^k(t) = \emptyset$ otherwise, and q^k is the profile of vacant seats at the beginning of Step k.

We prove by induction. Consider Step 1. Note that μ^1 is such that the smallest-index teacher among finest applications, say t, is assigned to her smallest-indexed feasible school. By Lemma 3, there exists a maximum size matching that coincides with μ^1 for the only teacher who has been considered by the end of Step 1.

Assume that the claim is true for any step before some Step k. Consider Step k. Let teacher t be the teacher considered in Step k.

Case 1: Suppose that t is critical for maximizing the size of (A, q). By Lemma 4, t is critical for maximizing the size of the reduced problem (A^k, q^k). Then, t is tentatively accepted to the smallest index school in $A(t)$ that has a vacant seat, say s. By the induction assumption, there exists a maximum size matching for the entire problem that coincides with μ^{k-1} for the teachers who have been considered by the end of Step $k-1$. By Lemma 3, there exists a maximum size matching for the reduced problem where t is assigned to s, say μ. Then, by Lemma 2, μ^{k-1} combined with μ, which coincides with μ^k for the teachers who have been considered by the end of Step k, is a maximum size matching for the entire problem (A, q).

Case 2: Suppose that t is not critical for maximizing the size of (A, q). By Lemma 4, t is not critical for maximizing the size of the reduced problem (A^k, q^k). By the induction assumption, there exists a maximum size matching for the entire problem that coincides with μ^{k-1} for the teachers who have been considered by the end of Step $k-1$.

Suppose that t is tentatively accepted to s because s has a vacant seat. By Lemma 3, there exists a maximum size matching for the reduced problem where t is assigned to s, say μ. Then, by Lemma 2, μ^{k-1} combined with μ, which coincides with μ^k for the teachers who have been considered by the end of Step k, is a maximum size matching for the entire problem (A, q).

Suppose that t is tentatively accepted to s by replacing another teacher t'. Since t is not critical for maximizing the size of the reduced problem (A^k, q^k), there exists a maximum size matching μ for the reduced problem where t is not matched to any school. Then, by Lemma 2, μ^{k-1} combined with μ, let us call it μ', is a maximum size matching for the entire problem (A, q). Now, consider the matching μ'' obtained from μ' by just replacing t with t' at school s. Note that this is feasible, μ'' is also a maximum size matching for the entire problem (A, q) and coincides with μ^k for the teachers who have been tentatively accepted by the end of Step k.

Suppose that t is rejected by s. Since t is not critical for maximizing the size of the reduced problem (A^k, q^k), there exists a maximum size matching μ for the reduced problem where t is not matched to any school. Then, by Lemma 2, μ^{k-1} combined with μ, let us call it μ', is a maximum size matching for the entire problem (A, q) and coincides with μ^k for the teachers who have been tentatively accepted by the end of Step k. □

Lemma 6. *Suppose that, at some step of the $HC(A)$ algorithm, a teacher who is critical in (A, q) is assigned to a school s. Then, no teacher is ever rejected, in particular t is never rejected, by s in the* *course of the $HC(A)$ algorithm.*

Proof. Suppose that t is critical in (A,q). Then, t is assigned a vacant seat at the first step t is considered, say in Step k at school $s \in A(t)$. We claim that no teacher is ever rejected, in particular t is never rejected, by s in the course of the algorithm. Clearly, no teacher is rejected from s until and including Step k. Suppose, towards a contradiction, that there is an earliest step, say Step k', at which a teacher is rejected from s. Then, the teacher t' who is considered at Step k' is not critical in (A,q) or in $(A^{k'},q^{k'})$, and t' replaces a teacher in s at Step k'. Since t' is not critical in $(A^{k'},q^{k'})$, by Lemma 2, there exists a maximum size matching for the entire problem (A,q), say μ, in which t' is not assigned to any school and t is assigned to s. But then, the matching obtained from μ by replacing t with t' in s is also a maximum size matching for the entire problem (A,q), contradicting that t is critical in (A,q). □

Lemma 7. *Suppose that $t \neq t'$. If t is critical in (A,q), then t is critical also in the reduced problem $(A_{-t'},q)$.*

Proof. Suppose, towards a contradiction, that t is not critical in $(A_{-t'},q)$. Then, there exists a maximum matching μ for the problem $(A_{-t'},q)$ in which t is not assigned to any school. Since t is critical in (A,q), μ is not a maximum size matching in (A,q). Then, there exists a list of teachers $t_1,\ldots,t_k$ and a school s such that t_1 is not assigned to any school in μ, all other teachers in the list are assigned to some schools in μ, and s has a vacant seat in μ, such that $\mu(t_i) \in A(t_{i-1})$ for each $i \in \{1,\ldots,k\}$ and $s \in A(t_k)$, i.e., there exists an *augmenting path* (Berge, 1957). Since μ is a maximum matching for the problem $(A_{-t'},q)$, $t_1 = t'$. Let μ' be the matching obtained from μ by assigning t' to s_1 and implementing the reassignment along the augmenting path. Now, observe that the size of μ' is one more than the size of μ, and μ' is a maximum size matching for the problem (A,q) (since (A,q) differs from $(A_{-t'},q)$ only by restoring the application of teacher t', their maximum size matchings can differ by at most one in size) where t is not assigned to any school, contradicting that t is critical in (A,q). □

Lemma 8. *HC is maximum-size.*

Proof. Directly follows from Lemma 5. □

Lemma 9. *HC^{Φ} is priority-respecting.*

Proof. If $HC_t(A) = \emptyset$, it must be that t has applied to each school in $A(t)$ in the course of the HC algorithm and got rejected from each of them. Note that whenever t is rejected by some $s \in A(t)$ at a step of the HC algorithm, all seats in s must be assigned to teachers with higher priorities at that step and thereafter. □

Lemma 10. *HC^{ϕ} is substitutable.*

Proof. Suppose that $t \neq t'$ and $HC_t(A) \neq \emptyset$. We want to show that $HC_t(A_{-t'}) \neq \emptyset$. Suppose that t is critical in (A, q). Then, by Lemma 7, t is critical also in $(A_{-t'}, q)$, and by Lemma 6, $HC_t(A_{-t'}) \neq \emptyset$.

Suppose that t is not critical in (A, q). Let $HC_t(A) = s$. Suppose, towards a contradiction, that $HC_t(A_{-t'}) = \emptyset$. By priority-respectingness, in $HC(A_{-t'})$, all seats in s are assigned teachers who have higher priority than t, and by Lemmas 6 and 7, all these teachers are not critical neither in (A, q) or in $(A_{-t'}, q)$.

Then, there exists a teacher t^* with $HC_{t^*}(A_{-t'}) = s$ such that t^* is assigned to a s' which has a lower index than s, in $HC(A)$, while t^* is rejected by s' in the $HC(A_{-t'})$ algorithm. Then, there exists a teacher t^{**} with $HC_{t^{**}}(A_{-t'}) = s'$ such that t^{**} is assigned to a school s'' that has a lower index than s, in $HC(A)$, while t^{**} is rejected by s' in the $HC(A_{-t'})$ algorithm. Continuing similarly, we reach a contradiction since the number of schools is finite. $\square$